– Rigid bodies

constrained   unconstrained

$\parallel^{el}$ to particles, we are looking for $[x]_t$

→ $[x]_t$ for "critical" points on the object (c.o.g, c.o.m., p.o.c) etc.

→ As in prev case define a state vector

$$y(t) = \begin{bmatrix} [x]_i^t \\ (v)_i^t \end{bmatrix} \quad i, = 1 - n \ \# \text{ of key points}$$

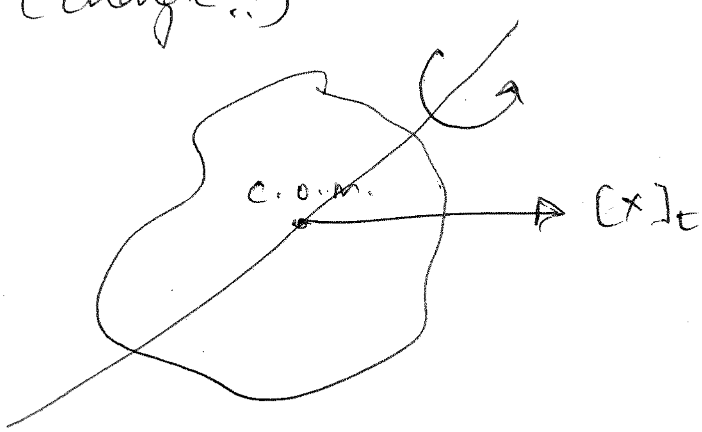
→ Assume $F(t)$ to be all forces acting on the particle

Then.

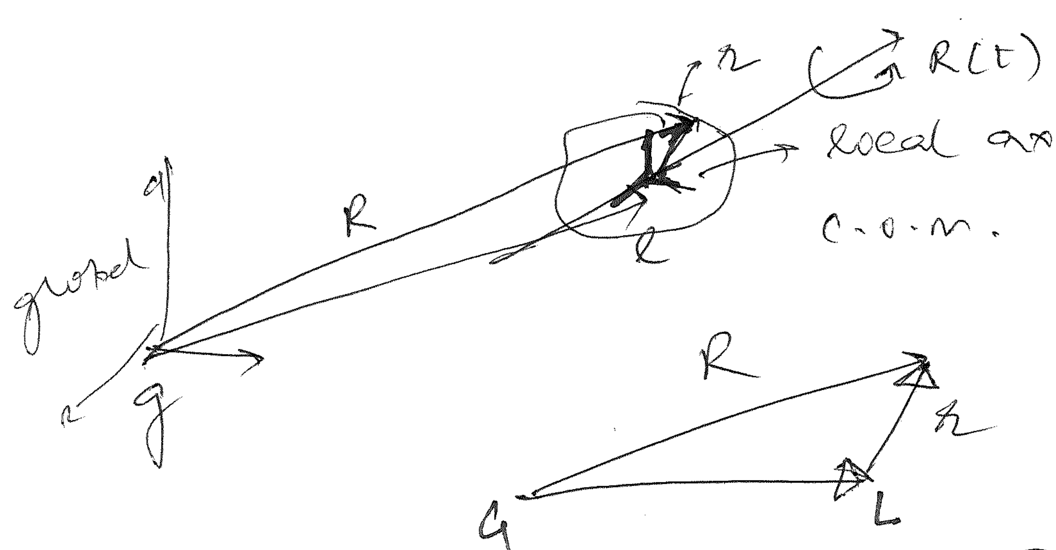
$$\frac{d}{dt} y(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(t)/m \end{pmatrix}$$

Note! upto here everything is ditto as for particles

What is the main complication for a rigid body?
(change!!)



$[x]_t$ is insufficient to describe the body's location in space. need $[R]_t$ too!



local axis, located at c.o.m.

$$\therefore [x]_t = [x(t), R(t)]$$

what about velocity?

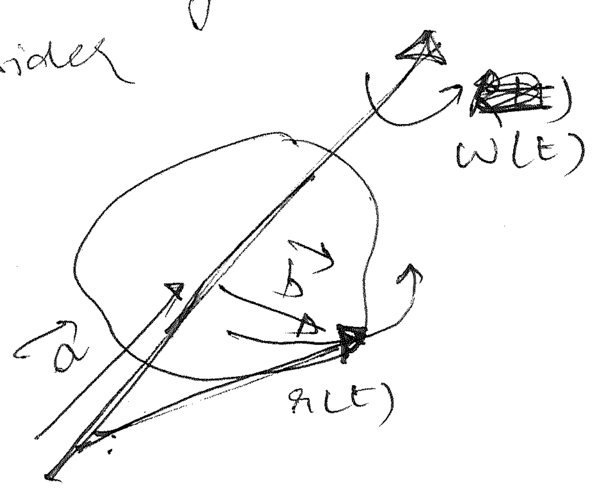
$$v(t) = \dot{x}(t) \quad (\text{simple!})$$

what about spinning? i.e. $\omega(t)$

(why spinning! remember c.o.m. condition)

Obviously $\omega(t) \neq \dot{R}(t)$

consider



→ $r(t)$ is some direction vector fixed on the body

→ then we would have

$$|\omega(t) \times b| = |\omega(t)|\,|b|$$

&

$$\dot{r}(t) = \omega(t) \times b$$

we also have

$$r(t) = \vec{a} + \vec{b} \quad \& \quad w(t) \times a = 0$$

$\therefore$

$$\dot{r}(t) = w(t) \times b \text{ or}$$

$$w(t) \times b + w(t) \times a$$

$$= \underbrace{w(t)}_{\text{magnitude}} \times \underbrace{\cancel{(b+a)} \rightarrow r(t)}_{\text{axis of rotation magnitude only!!}}$$

basically derivation of $v = r \times v$

|||ly

$$\dot{R} = \left[w(t) \times \begin{pmatrix} R_{xx} \\ R_{xy} \\ R_{xz} \end{pmatrix} \quad w(t) \begin{pmatrix} R_{yx} \\ R_{yy} \\ R_{yz} \end{pmatrix} \quad w(t) \begin{pmatrix} R_{zx} \\ R_{zy} \\ R_{zz} \end{pmatrix} \right]$$

an alternate form for $a \times b$, define operator $*$

such that

$$a * b = \overbrace{\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ a_y & a_x & 0 \end{pmatrix}}^{\text{[skew symmetric, } A^T = -A} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \Big(a \times b$$

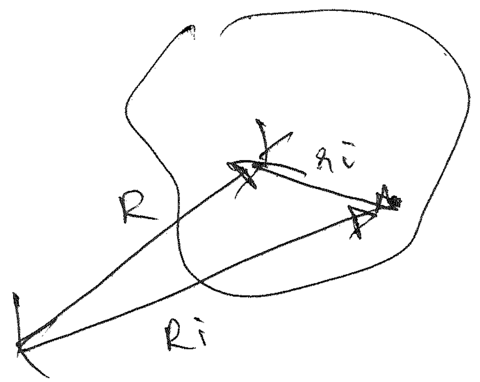
$\therefore$

$$\dot{R} = \left[w(t)* \begin{pmatrix} R_x \end{pmatrix} \quad w(t)* \begin{pmatrix} R_y \end{pmatrix} \quad w(t)* \begin{pmatrix} R_z \end{pmatrix} \right]$$

or

$$\dot{R}(t) = \omega(t) * \left[R(t) \right]$$

Another way of looking at it



$R_i = R + r_i$

$R_i = R + R' r_{io}$ [$R' \Rightarrow$ rotn matrix

$V_i = V + \frac{dR'}{dt} r_{io}$ | $R' R'^T = I$

or

$V_i = V + \underbrace{\frac{dR'}{dt} R'^T}_{\text{Skew symmetric matrix}} r_i$ [How did we make the jump?

or $V_i = V + \underbrace{\underbrace{\omega}_{\text{magnitude?}} \times \underbrace{r_i}_{\text{direction?}}}_{R}$

Other key properties?

Mass:-

$r_i(t) = R(t)\, r_{oi} + x(t)$ — x

$M = \sum_{i=1}^{N} m_i$

Convenient form for velocity!

— differentiate (*)

$$\dot{r}_i(t) = \overbrace{\omega(t) \times (r_i(t) - x(t))}^{\text{angular!}} + \underbrace{v(t)}_{\text{linear}}$$

C.O.M?

property $\quad \dfrac{\sum m_i r_{0i}}{M} = \vec{0}$

Torque?

$$\tau_i(t) = (r_i(t) - x(t)) \times F_i(t)$$

&

$$F(t) = \sum F_i(t)$$

Linear momentum

$$P = mv$$

$$P(t) = \sum \underbrace{m_i \dot{r}_i(t)}_{\text{angular + linear}}$$

$$P(t) = M\,v(t)$$ [How?]

& also

$$\boxed{\dot{P}(t) = \Sigma F(t)}$$

$M\ddot{x} = \Sigma F$

$\ddot{x} \propto$

$\dot{v} = \frac{\Sigma F}{M}$

<u>Angular momentum</u>

$$\boxed{\dot{L}(t) = \tau(t)}$$

$L(t) = I\,\omega(t)$

$I\alpha = \Sigma\tau$

$$I(t) = \sum m_i\,(r_i'^T r_i')\mathbb{1} - r_i' r_i'^T)$$

or

$$I_{body} = \sum m_i\,((r_{0i}^T\, r_{0i})\mathbb{1} - r_{0i}\, r_{0i}^T)$$

$$\boxed{I(t) = R(t)\, I_{body}\, R(t)^T}$$

Equations of Motion

$$Y(t) = \begin{bmatrix} x(t) \\ R(t) \\ P(t) \\ L(t) \end{bmatrix}$$

M
I_{body}
compute
$I(t)$ $w(t)$
& $v(t)$

but we know that

$$v(t) = \frac{P(t)}{M}, \quad I(t) = R(t) I_{body} R(t)^T$$

and $w(t) = I(t)^{-1} L(t)$

also
$$\frac{d}{dt} Y(t) = \begin{bmatrix} v(t) \\ w(t) * R(t) \\ f(t) \\ \tau(t) \end{bmatrix}$$

↳ Run Simulation

→ Init states $\left\{ \begin{array}{l} \text{mass}, I_{body}, I_{body}^{-1} \\ x, R, \underbrace{P, L}_{=0} \end{array} \right.$

↳ Compute other variables

$$v(t) = \frac{P(t)}{M}$$

$$I^{-1}(t)$$

& $\omega(t)$

↳ Compute force and torque

↳ Calculate

$$\cancel{v(t) = \frac{d}{dt}x(t)}$$

$$\frac{d}{dt}x(t) = v(t)$$

$$\dot{R}(t) = \omega(t) * R(t)$$

$$\frac{d}{dt}P(t) = F(t)$$

$$\frac{d}{dt}L(t) = \tau(t)$$

↳ Recalculate other variables and update states.

What forces to compute?

→ Gravity? = Mg

— acts @ c.o.g / c.o.m

→ Rotation free movement?

What is the torque calculation like?

$$\left[a(t) + \begin{bmatrix} \text{fulcrum} \\ \text{vector} \end{bmatrix} - x(t) \right] \times F_R$$