Robust price-setting newsvendor model with interval market size and consumer willingness-to-pay

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A B S T R A C T
We consider a price-setting newsvendor problem with partial information. The newsvendor does not know the price-dependent probability distribution of demand, but is able to estimate lower and upper limits of the market size and consumer willingness-to-pay. The objective is to minimize the maximum loss in expected profit, or minimax regret. We derive closed-form expressions for optimal quantity and price and identify managerial insights.

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1. Introduction

The classical newsvendor model has been commonly used in the literature to study the ordering problem for a short life-cycle product with uncertain demand (see e.g., Porteus, 1990). Recently we have seen a growing literature on the price-setting newsvendor problem to study the joint ordering and price decisions (see e.g., Petruzzi and Dada, 1999).

The traditional newsvendor models assume that a manager has perfect knowledge of the probability distribution of demand. However, for a new product with a short life-cycle and significantly innovative attributes, the manager may not be able to accurately estimate the demand probability distribution with little historical sales data or with subjective forecasting methods (Perakis and Roels, 2008). The manager may incur a significant loss in profit if he or she makes quantity and price decisions based upon an inaccurate probability distribution of demand. For example, when IBM introduced a novel cursor control in the keyboard of its Thinkpad computer, the managers had difficulty believing the enthusiastic reactions to this new feature in early focus groups. As a result, IBM underestimated demand and suffered shortages for more than a year (Fisher, 1997). Similarly, when Dell Computer Corporation planned to introduce its Olympic computers, they misjudged customers’ preferences for the technology that far exceeded anything in the industry at that time. Due to the lack of consumer interests in the new technological features, Dell cancelled the Olympic product line and admitted they made a mistake (Dell and Fredman, 2000). These examples illustrate some challenges of accurately estimating the price-dependent probability distribution of demand for new and innovative products and the need for robust newsvendor decisions that perform well under various demand scenarios.

To overcome the limitation of traditional newsvendor models, one alternative approach is to analyze a newsvendor problem with an objective of minimizing the maximum loss in expected profit due to incomplete information on the probability distribution of demand (see e.g., Perakis and Roels, 2008). The minimax regret formulation originated from Savage (1951) is a useful decision-making framework on two levels. First, minimax regret solutions provide guidance for decisions in an environment with limited information on relevant probability distributions. Second, the minimax regret computation lends insight into the value of additional market information; the objective function gives an upper bound on the gain in expected profit from perfect knowledge of the probability distribution.

Our paper is based upon the minimax regret approach that requires minimal information on a market’s response to an innovative product—we require lower and upper limits on the number of consumers who will consider purchasing the product during the selling season (i.e., market size interval) and the price that the product can command (i.e., willingness-to-pay interval). The use of interval data...
for decision-making under uncertainty is widely used in the literature (e.g., Vairaktarakis, 2000; Perakis and Roels, 2008; Lan et al., 2008; Jang et al., 2011) and also is feasible in practice (see e.g., Fisher and Raman, 1996; Hammond and Raman, 1996). Lin and Ng (2011), for example, describe how the fashion retail chain ERL uses interval data to support order quantity decisions. We identify ordering and pricing solutions that are robust over the set of probability distributions that are consistent with the interval data. In particular, we identify robust solutions that minimize the maximum loss in expected profit due to ignorance of the price-dependent probability distribution of demand, or minimax regret (Savage, 1951).

The first main contribution of our paper relates to the definition, solution, and interpretation of a single-product simultaneous ordering and pricing minimax regret problem. As far as we know, there is no previous research on the price-setting newsvendor problem within the minimax regret framework. For the problem, we derive closed form solutions for the optimal decisions and the minimax regret. We find a dichotomy in the optimal pricing and ordering rules. A product can be classified as either high-profit or low-profit depending upon the relative spreads between clearance price, purchase cost, and minimum consumer valuation. For high-profit product, the ordering and pricing decisions essentially become independent and can be made separately, and rather simple decision rules apply. The optimal decision rules are more complex for low-profit product due to the linkage between price and quantity decisions.

The second main contribution relates to the robustness of the traditional newsvendor model. We numerically evaluate the robustness of the newsvendor model when there is little basis for estimating relevant probability distributions. Our findings are generally consistent with the principle of insufficient reason (Bernoulli, 1954): when there is no basis for estimating probabilities, all outcomes should be considered equally likely.

In the next section we explain how our work relates to the literature. Section 3 presents our model and results. Section 4 concludes with a summary. All proofs, intermediate propositions, and technical derivations are located in the Appendix of proofs and derivations.

2. Related literature

There is a large body of literature directed at the problem of setting quantity and/or price to maximize expected profit when there is no chance for placing replenishment orders during the selling season. The newsvendor literature has focused on the quantity decision (e.g., Porteus, 1990 for an overview) and the revenue management literature has focused on the pricing decision (e.g., see Talluri and van Ryzin, 2004 for an overview). There is a recent and growing literature on quantity and price decisions (e.g., Dana and Petruzzi, 2001; Granot and Yin, 2008; Petruzzi and Dada, 1999, 2001, 2002; Roels, 2008; Wang, 2006).

Scarf (1958) and Gallego and Moon (1993) relax the assumption that the probability distribution of demand is known by the newsvendor. They assume that only the mean and variance of demand are known. The objective is to determine the order quantity that maximizes the minimum expected profit over all possible demand distributions. In addition, a number of the researchers have extended this line of research (e.g., Alfreed and Elmorra, 2005; Gallego et al., 2001; Moon and Choi, 1995, 1997, 1998; Moon and Gallego, 1994; Moon and Silver, 2000; Ouyang and Chang, 2002; Silver and Moon, 2001, Wu and Ouyang, 2001; Yue et al., 2007).

Kasugai and Kasegai (1961), Morris (1959), and Vairaktarakis (2000) analyze a newsvendor order quantity problem where the objective is to minimize the maximum loss in expected profit due to ignorance of realized profit. Mostard et al. (2005), Perakis and Roels (2008), and Yue et al. (2006) also analyze a newsvendor ordering problem and the objective is to minimize the maximum loss in expected profit due to incomplete information on the distribution of demand. Lin and Ng (2011) analyze a multi-market newsvendor ordering problem with the objective of minimizing the maximum loss in expected profit when only the intervals of market demands are known.

We use the same objective as Lin and Ng (2011), Mostard et al. (2005), Perakis and Roels (2008), and Yue et al. (2006). We extend this work to problems with pricing and order quantity decisions. As a consequence, we model demand through consumer valuation. This approach is frequently used in price-setting models because it allows one to represent price-dependent demand in terms of an underlying model of consumer behavior (e.g., see Lilian et al., 1992 or Phillips, 2005).

Our work is also related to the growing research stream of robust revenue management (e.g., Ball and Queyranne, 2009; Birbil et al., 2009; Eren and Maglara, 2006; Gueller, 2014; Lan et al., 2007, 2008; Lee and Hsu, 2011; Perakis and Roels, 2010). While this research stream focuses on the robust booking (or capacity) control problem for a firm selling fixed inventory over time with no-or-limited demand information, our research focuses on the robust ordering and pricing decisions for a new product with limited consumer valuation and market size information.

3. Model and results

A retailer wishes to order and price a new product for the upcoming selling season. There are $M$ consumers who will visit the store and consider purchasing one unit of the product during the selling season. The retailer knows that the size of the market, is at least $m_{\text{min}}$ and no more than $m_{\text{max}}$, i.e., $M \in \Lambda = [m_{\text{min}}, m_{\text{max}}]$. The case where the retailer knows the market size corresponds to $m_{\text{min}} = m_{\text{max}}$.

We consider a heterogeneous market demand model that treats $V$ as the valuation of a randomly selected consumer. Only the interval containing $V$ is known to the decision maker. We define willingness-to-pay function $W(p) = P[V < p]$, which is unknown to the decision maker. Specifically, $W(p)$ defines the valuation distribution among the population of $M$ consumers, and thus $1 - W(p) = P[V \geq p]$ is the fraction of consumers who will purchase the product at retail price $p$. For a given market size $M$, demand is $D(p) = M[1 - W(p)]$, which is a deterministic function that is nonincreasing in $p$. Deterministic price-demand functions are widely used in marketing and economic models1

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1 We also consider an alternative demand model that assumes that consumers are homogeneous in their valuation, and will each purchase one unit of the product if valuation $V$ is greater than or equal to the retail price $p$ (Lazear, 1986 uses the same revenue model; see §I.D in his paper). Since we obtain the same results, in this paper we focus only on the heterogeneous demand model.
The unit purchase cost is $c$. Any product that remains unsold by the end of the season is sold at unit salvage value $s$. As in the classic newsvendor model, we limit consideration to the more interesting case where salvage value is less than cost (i.e., $s < c$); the ordering decision is trivial when $s \geq c$. The retailer’s profit as a function of order quantity $q$ and retail price $p$ is 
$$\Pi(p, q) = p \min \{D(p, q)\} + s \max \{0, q - D(p)\} - cq,$$
and for a given market size $M$ and willingness-to-pay function $W$, the retailer’s profit is 
$$\Pi_{MW}(p, q) = (p - s)\min \{M[1 - W(p)], q\} - (c - s)q.$$

The retailer does not know the willingness-to-pay function $W(p)$, but she does know that the minimum valuation is no less than $v_{\min}$ and the maximum valuation is no more than $v_{\max}$. We assume that $v_{\min} \geq s$ (i.e., the minimum valuation is not less than the end-of-season clearance price) and that $v_{\min} \geq c$, e.g., consumers will purchase the product if the retailer sells at cost.

We let $\Omega$ denote the set of the unknown willingness-to-pay functions $W(p)$ with support bounded by known interval $[v_{\min}, v_{\max}]$, i.e.,
$$\Omega = \{W : W(v_{\min}) = 0, W(p + \varepsilon) \geq W(p) \ \forall p, \varepsilon > 0, W(p) = 1 \ \forall p > v_{\max}\}.$$

The retailer’s problem is to minimize her maximum regret due to market ignorance as manifested in incomplete knowledge of $M$ and $W$. The decision sequence is as follows:

1. Retailer selects price $p$ and order quantity $q$.
2. Nature selects $W \in \Omega$, $M \in \Lambda$, price $\hat{p}$, and order quantity $\hat{q}$ to maximize the retailer’s loss in profit.

The retailer’s problem can be expressed as
$$\rho^* = \min_{p, \hat{p}} \max_{M \in \Lambda, W \in \Omega} \left\{ \max_{p, q} \Pi_{MW}(\hat{p}, \hat{q}) - \Pi_{MW}(p, q) \right\} \quad (1)$$

The first term in the brackets in (1) is the optimal profit if the market size is $M$ and the willingness-to-pay function is $W$. The value of $\rho^*$ is the minimax regret, or loss in profit due to market ignorance. Alternatively, $\rho^*$ is the maximum increase in profit if market research is able to expose $M$ and $W$, and as such, provides insight into drivers of the value of additional market research prior to finalizing price and quantity decisions. The following proposition characterizes the optimal price and quantity decisions and minimax regret

**Proposition 1.** The optimal price, quantity, and minimax regret are as follows:

- if $(v_{\min} - c) \geq c - s$, then
  $$p^* = \begin{cases} v_{\min}, & \text{if } v_{\max} \leq v_{\min} + c - s \\ v_{\min}, & \text{if } v_{\min} + c - s \leq v_{\max} \leq 2v_{\min} - s \\ s + \frac{(c - s)}{v_{\min} - v_{\max}}, & \text{if } v_{\max} \geq 2v_{\min} - s \\ \left(1 - \frac{v_{\min} - c}{v_{\max} - v_{\min}}\right)v_{\min} + \frac{(v_{\max} - v_{\min})}{v_{\max} - v_{\min}}v_{\max}, & \text{if } v_{\max} \leq v_{\min} + c - s \end{cases}$$
  $$q^* = \begin{cases} v_{\max}, & \text{if } v_{\min} + c - s \leq v_{\max} \leq 2v_{\min} - s \\ m_{\max}, & \text{if } v_{\max} \geq 2v_{\min} - s \\ m_{\max} - \frac{(v_{\min} - c)}{v_{\min} - v_{\max}}, & \text{if } v_{\max} \leq v_{\min} + c - s \end{cases}$$
  $$\rho^* = \begin{cases} v_{\max} - v_{\min}m_{\max}, & \text{if } v_{\min} + c - s \leq v_{\max} \leq 2v_{\min} - s \\ \frac{1}{2}(v_{\max} - s)m_{\max}, & \text{if } v_{\max} \geq 2v_{\min} - s \end{cases}$$
- if $(c - s)/4 \leq v_{\min} - c \leq c - s$, then
  $$p^* = \begin{cases} v_{\min}, & \text{if } v_{\max} \leq v_{\min} + c - s \\ v_{\min}, & \text{if } v_{\min} + c - s \leq v_{\max} \leq v_{\min} + 2[(v_{\min} - c)(c - s)]^{1/2} \\ s + \left(\frac{c - s}{v_{\min} - v_{\max}}\right)^{1/2}(v_{\max} - s), & \text{if } v_{\min} + 2[(v_{\min} - c)(c - s)]^{1/2} \leq v_{\max} \leq 4c - 3s \\ s + \frac{(c - s)}{v_{\min} - v_{\max}}, & \text{if } v_{\max} \geq 4c - 3s \\ \left(1 - \frac{v_{\min} - c}{v_{\max} - v_{\min}}\right)v_{\min} + \frac{(v_{\max} - v_{\min})}{v_{\max} - v_{\min}}v_{\max}, & \text{if } v_{\max} \leq v_{\min} + c - s \end{cases}$$
  $$q^* = \begin{cases} m_{\max}, & \text{if } v_{\min} + c - s \leq v_{\max} \leq v_{\min} + 2[(v_{\min} - c)(c - s)]^{1/2} \\ \left(\frac{(v_{\max} - s)}{v_{\min} - v_{\max}}\right)^{1/2} - 1)m_{\max}, & \text{if } v_{\min} + 2[(v_{\min} - c)(c - s)]^{1/2} \leq v_{\max} \leq 4c - 3s \\ m_{\max}, & \text{if } v_{\max} \geq 4c - 3s \\ \left(\frac{v_{\min} - c}{v_{\max} - v_{\min}}\right)(c - s)(m_{\max} - m_{\min}) + (v_{\max} - v_{\min})m_{\min}, & \text{if } v_{\max} \leq v_{\min} + c - s \end{cases}$$
  $$\rho^* = \begin{cases} (v_{\max} - v_{\min})m_{\max}, & \text{if } v_{\min} + c - s \leq v_{\max} \leq v_{\min} + 2[(v_{\min} - c)(c - s)]^{1/2} \\ 2(c - s)^{1/2}(v_{\max} - s)^{1/2} - (c - s)^{1/2}m_{\max}, & \text{if } v_{\min} + 2[(v_{\min} - c)(c - s)]^{1/2} \leq v_{\max} \leq 4c - 3s \\ \frac{1}{2}(v_{\max} - s)m_{\max}, & \text{if } v_{\max} \geq 4c - 3s \end{cases}$$
if $v_{\text{min}} - c \leq (c-s)/4$, then

$$p^* = \begin{cases} v_{\text{min}}, & \text{if } v_{\text{max}} \leq s+(c-s)(1+1-2a+ab)^{1/2}/a^2 \\ s + \frac{c-s}{v_{\text{max}}-s}, & \text{if } v_{\text{max}} \leq s+(c-s)(1+1-2a+ab)^{1/2}/a^2 \leq 4c-3s \\ C, & \text{if } v_{\text{max}} \geq 4c-3s \end{cases}$$

$$q^* = \begin{cases} v_{\text{min}}, & \text{if } v_{\text{max}} \leq s+(c-s)(1+1-2a+ab)^{1/2}/a^2 \leq 4c-3s \\ \frac{(v_{\text{max}}-c)^{1/2}}{2} - 1, & \text{if } v_{\text{max}} \leq s+(c-s)(1+1-2a+ab)^{1/2}/a^2 \leq 4c-3s \\ \max, & \text{if } v_{\text{max}} \geq 4c-3s \end{cases}$$

where

$$a = \frac{m_{\text{min}}}{m_{\text{max}}} + \left(1 - \frac{m_{\text{min}}}{m_{\text{max}}} \right) \frac{c-s}{v_{\text{min}}-s}$$

$$b = a + \frac{m_{\text{min}}}{m_{\text{max}}} \frac{v_{\text{min}}-c}{c-s} = \left(1 - \frac{m_{\text{min}}}{m_{\text{max}}} \right) \frac{c-s}{v_{\text{min}}-s} + \frac{m_{\text{min}}}{m_{\text{max}}} \frac{v_{\text{min}}-s}{c-s}.$$  

3.1. Interpretations of optimal decisions and the value of additional information

In this section we focus on exposing the rules and intuition that underlie the expressions in Proposition 1. We will show that the optimal decisions have the following basic form

\[ \text{decision} = \text{base} + \% \times \text{gross at risk}, \]

and the minimax regret has the following basic form

\[ \text{regret} = \text{max loss due to poor price} + \text{max loss due to poor order quantity} \]

\[ = \text{$/unit at risk} \times \text{quantity} + \text{$/unit at quantity at risk} \]

To facilitate our explanations, we reinterpret the expressions of Proposition 1 for the two extreme cases of ignorance in minimum market size: (1) $m_{\text{min}} = m_{\text{max}}$ and (2) $m_{\text{min}} = 0$ (see Tables 1–3). The interpretations of results for the alternative cases of $m_{\text{min}} \in (0, m_{\text{max}})$ take a middle ground, combining the insights that come from the extremes of $m_{\text{min}} \in (0, m_{\text{max}})$.

At price $p$, the newsvendor ratio is $\alpha(p) = (p-c)/(p-s)$. We let $\eta(p) = 1 - \alpha(p)$, which is the complement of the newsvendor ratio (CNR). The retailer will never set price higher than $v_{\text{max}}$ and thus $\eta(v_{\text{max}})$ is a lower bound on the complement of the newsvendor ratio (CNR$_{10}$). Note that $\eta(p) = (p-c)/(c-s)$, which we call the under-to-over ratio because $p-c$ is the unit loss from under ordering and $c-s$ is the unit loss from over ordering. We refer to $(v_{\text{max}}-c)/(v_{\text{min}}-s)$ as the gain-to-loss ratio because $v_{\text{max}}-c$ is an upper limit on the unit profit margin (gain) and $v_{\text{min}}-s$ is a lower limit on the unit profit loss from over pricing.

The superscripted numbers in Tables 1–3 identify exceptions to the simple relationships in the tables:

1. If $p^* < v_{\text{min}}$, then $p^* = v_{\text{min}}$.
2. If $p^* > m_{\text{max}}$, then $q^* = m_{\text{max}}$.
3. If $p^* > c + (v_{\text{min}}-c)/(c-s)/0.5$, then $p^* = v_{\text{min}}$ and $q^* = m_{\text{max}}$.
4. If $0.25(c-s) \leq v_{\text{min}}-c$ and $p^* < c + (v_{\text{min}}-c)/(c-s)/0.5$, then $p^* = v_{\text{min}}$ and $q^* = \min(1, (v_{\text{max}}-c)/(v_{\text{min}}-s))m_{\text{max}}$; or if $0.25(c-s) \geq v_{\text{min}}-c$ and $p^* < 2v_{\text{min}}-c$, then $p^* = v_{\text{min}}$ and $q^* = \min(1, (v_{\text{max}}-c)/(v_{\text{min}}-s))m_{\text{max}}$.
5. If the optimal price from Table 1 is $v_{\text{min}}$, then ‘\(^\prime\)’ is replaced with ‘\(\leq\)’.

Clearly, the retailer will never set price below the minimum valuation $v_{\text{min}}$ or set the quantity above the maximum possible demand $m_{\text{max}}$. This observation is reflected in notes 1 and 2. Notes 3 and 4 pertain to discontinuities in the optimal decision. Note 5 reflects that fact that over-pricing error disappears when $p^* = v_{\text{min}}$. We will expand on notes 3–5 below.

From Tables 1 and 2, we see that the optimal price and quantity formulas can be divided into two groups defined by the relationship between $v_{\text{min}}-c$ and $c-s$. We refer to a product with $v_{\text{min}}-c \geq c-s$ as a high-profit product because the minimum valuation is relatively high, and we refer to product with $v_{\text{min}}-c \leq c-s$ as a low-profit product because the minimum valuation is relatively low. Note that the newsvendor ratio is at least 50% when high-profit product price is set to $v_{\text{min}}$ and the newsvendor ratio is at most 50% when low-profit product price is set to $v_{\text{min}}$.

For high-profit product, we see a particularly simple pricing rule: set price to split the difference between the salvage value and the maximum valuation. The quantity decision rule is also relatively simple. Indeed, once $v_{\text{min}}-c$ is above the threshold $c-s$ that delineates low- and high-profit product, a linkage between price and quantity decisions disappears; the retailer sets the price as if quantity is exogenous and sets the quantity to account for solely for ignorance of market size.\(^2\) If there is no market size ignorance as in Table 1 (i.e., $m_{\text{min}} = m_{\text{max}}$), the quantity is set to the market size (and maximum demand) $m_{\text{max}}$. Alternatively, when market size ignorance is high as in

\(^2\) Results for optimal decisions when either price or quantity are exogenous are relatively straightforward to derive, and are available upon request to the corresponding author.
Table 2 (i.e., $m_{\text{min}}=0$), the quantity is set to a percentage of its upper limit, where the percentage is the gain-to-loss ratio $(v_{\text{max}}-c)/(v_{\text{min}}-s)$. The structure of the optimal price and quantity formulas continue to apply when $0 < m_{\text{min}} < m_{\text{max}}$. A retailer who is ordering and pricing a low-profit product may begin by examining the value of $\bar{p} = s + m_{\text{max}}(v_{\text{max}}-s)$. If $\bar{p}$ is close to $v_{\text{min}}$ (see notes 3 and 4 for exact conditions), then the retailer sets $p^* = v_{\text{min}}$ and thereby avoids any risk of over-pricing. Alternatively, if $\bar{p}$ is very large, then the retailer uses the simple pricing and ordering rules that apply for high-profit product. Otherwise $p^* = \bar{p}$ and the under-to-over ratio $(p^*-c)/(c-s)$ gives the fraction of $m_{\text{max}}$ to order.

Table 3 shows that there two equivalent perspectives of minimax regret. On one hand we have

$$\rho^* = (v_{\text{max}}-p^*)m_{\text{max}} + (p^*-c)(m_{\text{max}}-q^*),$$

which can be viewed as the loss of under-pricing and under-ordering in a hot market (i.e., the market size is $m_{\text{max}}$ and all consumers have valuation $v_{\text{max}}$). For example, when market size and valuation take their maximum values (e.g., hot market), we could have priced at $v_{\text{max}}$ instead of $p^*$ thereby avoiding the under-pricing loss of $(v_{\text{max}}-p^*)m_{\text{max}}$. Similarly, at our price $p^*$ we could have ordered (and sold) $m_{\text{max}}$ instead of $q^*$ thereby avoiding the under-ordering loss of $(p^*-c)(m_{\text{max}}-q^*)$. On the other hand, we also have

$$\rho^* = (p^*-c)m_{\text{max}} + (c-s)q^*,$$

which can be viewed as the loss of over-pricing and over-ordering in a cold market (i.e., the market size is $m_{\text{max}}$ and all consumers have valuation $p^*$ that is infinitesimally less than $p^*$). For example, in a cold market, we could have priced at $p^*-c$ and sold out instead of $p^*$ and selling nothing thereby avoiding an over-pricing loss of $(p^*-c)m_{\text{max}}$. Similarly, at our price $p^*$, we could have ordered nothing instead of $q^*$ thereby avoiding an over-ordering loss of $(c-s)q^*$. Thus, we see that optimal pricing and quantity decisions are set to equalize the maximum loss due to pricing too low and ordering too little with the maximum loss due to pricing too high and ordering too much. Of course, if $p^* = v_{\text{min}}$ then there is no possibility for a loss due to over-pricing, which explains note 5: if $p^* = v_{\text{min}}$, then $\rho^* \leq (p^*-c)m_{\text{max}} + (c-s)q^*$ (see Proposition 1 for the exact expression for $\rho^*$ when $p^* = v_{\text{min}}$).

3.2. Robustness of the newsvendor model

The traditional newsvendor model, which requires knowledge of the probability distribution of demand at various prices, is a well-established decision-making framework. Managers may use this model to develop quantity and/or price decisions even when there is little basis for estimating the price-dependent probability distribution of demand. As in Perakis and Roels (2008), we examine the consequences of such an approach.

The setting is one in which the manager can accurately estimate the lower and upper limits of consumer valuation and market size, but has little basis for gauging the probability distribution. A manager applying the newsvendor model makes an assumption on the probability distribution of consumer valuation (which determines the willingness-to-pay function $W(p)$) and the probability distribution of the market size $M$, which yields a price-dependent probability distribution of demand.

We numerically evaluate the regret ratio for the case of known market size (i.e., $M=m_{\text{min}}=m_{\text{max}}$) and a distribution of consumer valuation that is uniform, logistic, normal, exponential, or beta. The uniform valuation distribution results in a linear price-demand function and the logistic valuation distribution results in a logit price-demand function, both of which are common in the literature and in practice (Phillips, 2005). The uniform, normal, and exponential distributions have been used in Perakis and Roels (2008) and the beta distribution has been used in Lin and Ng (2011) to test the robustness of the newsvendor models based upon the minimax regret approach.

We use numerical search to identify the price and order quantity that maximize expected profit $(p^*, q^*)$ and we compute the regret ratio $(\rho(p^*, q^*)/\rho^*)$ of the optimal solution. An expression for the maximum regret as a function of $p$ and $q$ ($\rho(p, q)$) is given in Lemma 1 in the Appendix of proofs and derivations. Fig. 1 displays the regret ratio curves as a function of valuation range at parameter values $v_{\text{min}}=100$, $c=50$, and $s=25$, which corresponds to the case of high-profit product in Table 1. Fig. 2 displays the regret ratio curves as a function of valuation range at parameter values $v_{\text{min}}=100$, $c=75$, and $s=25$, which corresponds to the case of low-profit product in Table 1.

When the product is a high-profit product, we see that the newsvendor ordering and pricing model is generally more robust when the newsvendor assumes a uniform distribution or a beta distribution with $\alpha=\beta=0.5$ (note that the uniform distribution is a special case of the beta distribution with $\alpha=\beta=1$). For example, the maximum regret ratio with the uniform distribution and the beta distribution with $\alpha=\beta=0.5$ is about 1.3 but is at least larger than 1.4 with other distributions for the case of high-profit product. However, when the product is a low-profit product, we see that the newsvendor ordering and pricing model is generally more robust when the newsvendor

![Fig. 1. Plots (on a logarithmic scale) of the regret ratio as a function of the range of valuation under the uniform valuation distribution, the logistic and normal valuation distributions with the mean and standard deviation ($\mu, \sigma$) satisfying $[v_{\text{min}}, v_{\text{max}}]=([2\mu-\sigma, 2\mu+\sigma])$, the exponential valuation distribution shifted by $v_{\text{min}}$ and with $v_{\text{max}}$ corresponding to the 95th percentile, and the beta valuation distributions with various shape parameters $(\alpha, \beta)$. Parameter values are $v_{\text{max}}=100$, $c=50$, $s=25$, e.g., a high-profit product.](image-url)
In this paper we define and analyze the ordering and pricing problem faced by a retailer of a new product with a short life-cycle and significantly innovative attributes. We derive rules for ordering and pricing when only intervals of the market and consumer lower and upper bound of willingness-to-pay are known. The rules are relatively simple and intuitively appealing—both the ordering and the pricing rules can be interpreted as a relevant base inflated by some percentage of a relevant value at risk, and they strike a balance between under ordering and over ordering in a hot market and over ordering and pricing in a cold market. The decision rules are optimal under the objective of minimizing the maximum loss in expected profit due to market ignorance. In sum, the rules provide guidance for ordering and pricing fashion product and illuminate how various factors influence the value of reducing market ignorance.
The max regret for PA1 is

\( \rho^* = \max_{\mathcal{A}, \mathcal{W} \in \Omega} \{ p \} \)

Due to poor pricing

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<th>( \rho^* )</th>
<th>Due to poor ordering</th>
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<td>( \rho^* = q )</td>
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</table>

Close examination of the optimal decision rules shows that there exists a dichotomy in the optimal solutions for the joint price/quantity problem. Depending on the setting, the expressions for optimal solutions are either rather simple or rather complex. Specifically, a product can be classified as either high-profit or low-profit depending upon the relative spreads between clearance price, purchase cost, and minimum consumer valuation. For high-profit product, the ordering and pricing decisions essentially become independent and rather simple decision rules apply. The optimal decision rules are more complex for low-profit product due to the linkage between price and quantity decisions.

We also extend our understanding of the robustness of the newsvendor decision-making framework when there is little basis for estimating relevant probability distributions. Previous research has examined the robustness of newsvendor decision-making framework for the ordering problem. A main result is that the newsvendor optimal order quantity minimizes the maximum regret if uniform demand is assumed. The result is consistent with the principle of insufficient reason: when there is no basis for estimating probabilities, all outcomes should be considered equally likely. We numerically examine the robustness of the price-setting newsvendor model. Our results generally support the principle of insufficient reason, though it is less compelling; in some instances the newsvendor framework with a uniform assumption leads to the same decisions as the minimax regret framework, in other instances it does not, and the increase in maximum regret can be significant.

Appendix of proofs and derivations

Before we prove Proposition 1, we introduce two lemmas that are intermediate results used in the proof of Proposition 1 and our numerical analysis in Section 3.2. Lemma 1 gives the maximum regret as a function of \( p \) and \( q \) relative to Nature. Lemma 2 gives optimal price and the maximum regret as a function \( q \) relative to Nature. Nature is free to set both price and quantity. The problems are

\[ \text{PA1: } \rho^*(p, q) = \max_{M \in \mathcal{A}, W \in \Omega} \left\{ \max_{p \in \mathcal{A}, q \in \mathcal{W}} (E_W[I(I(p, q))] - E_W[I(p, q)]) \right\} \]

\[ \text{PA2: } \rho^*(q) = \min_p \max_{M \in \mathcal{A}, W \in \Omega} \left\{ \max_{p \in \mathcal{A}, q \in \mathcal{W}} (E_W[I(I(p, q))] - E_W[I(p, q)]) \right\} \]

**Lemma 1.** The max regret for PA1 is

\[ \rho^*(p, q) = \begin{cases} 
(V_{\text{max}} - c)m_{\text{max}} - (p - c)q, & \text{if } p = v_{\text{min}} \text{ and } q \leq \min\{m_{\text{min}}, q\} + \frac{(V_{\text{max}} - c)}{p - s}(m_{\text{max}} - m_{\text{min}}) \\
(V_{\text{max}} - c)m_{\text{min}} - (p - s)\min\{m_{\text{min}}, q\} + (c - s)q, & \text{if } p = v_{\text{min}} \text{ and } q \geq \min\{m_{\text{min}}, q\} + \frac{(V_{\text{max}} - c)}{p - s}(m_{\text{max}} - m_{\text{min}}) \\
(V_{\text{max}} - c)m_{\text{max}} - (p - c)q, & \text{if } p > v_{\text{min}} \text{ and } q \leq \frac{(V_{\text{max}} - c)}{p - s}m_{\text{max}} \\
(p - c)m_{\text{max}} + (c - s)q, & \text{if } p > v_{\text{min}} \text{ and } q \geq \frac{(V_{\text{max}} - c)}{p - s}m_{\text{max}} 
\end{cases} \]

**Proof.** For a given \( M \in \mathcal{A}, W \in \Omega \), and \( \hat{p} \), let \( q^0 = \arg \max_{q} (E_W[I(I(\hat{p}, q))] \) where \( E_W[I(p, q)] = (p - s)\min\{1 - W(p)M, q\} - (c - s)q \). Note that \( q^0 \in [0, 1 - W(p)M] \). From \( p \geq v_{\text{min}} \), it follows that \( q^0 = 1 - W(\hat{p})M \). Thus, the problem can be restated as

\[ \rho^*(p, q) = \max_{M \in \mathcal{A}, W \in \Omega} \left\{ \max_{\hat{p}} (E_W[I(I(\hat{p}, [1 - W(\hat{p})M])] - E_W[I(p, q)]) \right\}. \]

Interchanging the order of maximization we get

\[ \rho^*(p, q) = \max_{\hat{p}} R(\hat{p}) \]

where

\[ R(\hat{p}) = \max_{M \in \mathcal{A}, W \in \Omega} (E_W[I(I(\hat{p}, [1 - W(\hat{p})M])] - E_W[I(p, q)]) = \max_{M \in \mathcal{A}, W \in \Omega} (\hat{p} - c)[1 - W(\hat{p})M - (p - s)\min\{1 - W(p)M, q\} + (c - s)q]. \]

If \( \hat{p} < p \), then \( R(\hat{p}) \) is obtained with \( W = 1 - W(p) = 0 \), and \( M = m_{\text{max}} \), i.e.,

\[ R(\hat{p}; \hat{p} < p) = (\hat{p} - c)m_{\text{max}} + (c - s)q. \]
If \( \hat{p} \geq p = v_{\min} \) (which implies \( W(\hat{p}) \geq W(p) = 0 \)), then for any given \( M \in A, R(\hat{p}) \) is obtained with \( W \) satisfying either \( 1 - W(p) = 1 - W(\hat{p}) = 1, \) i.e.,
\[
R(\hat{p}|M, \hat{p} \geq p = v_{\min}) = (p - c)M - (p - s)\min\{M, q\} + (c - s)q.
\]
Alternatively, if \( \hat{p} \geq p > v_{\min} \) (which implies \( W(\hat{p}) \geq W(p) \)), then for any given \( M \in A, R(\hat{p}) \) is obtained with \( W \) satisfying either \( 1 - W(p) = 1 - W(\hat{p}) = 0, \) i.e.,
\[
R(\hat{p}|M, \hat{p} \geq p > v_{\min}) = \max\{(p - c)M - (p - s)\min\{M, q\} + (c - s)q, (c - s)q\}.
\]
Observe that the only difference between \( R(\hat{p}|M, \hat{p} \geq p = v_{\min}) \) and \( R(\hat{p}|M, \hat{p} \geq p > v_{\min}) \) is the inclusion of a \((c - s)q\) term under the max operator. We know that \( \rho^*(p, q) \) is not smaller than \((c - s)q\), i.e.,
\[
\rho^*(p, q) = \begin{cases} 
(v_{\max} - c)m_{\max} - (p - c)q & \text{if } p > v_{\min} \text{ and } q \leq \left(\frac{v_{\max} - p}{c - s}\right)m_{\max} \\
(p - c)m_{\max} + (c - s)q & \text{if } p > v_{\min} \text{ and } q \geq \left(\frac{v_{\max} - p}{c - s}\right)m_{\max}
\end{cases}
\]
(the second expression clearly dominates \((c - s)q\), and the inequality on \( q \) ensures that the first expression, \((v_{\max} - c)m_{\max} - (p - c)q = (v_{\max} - c)m_{\max} - (p - s)q + (c - s)q\), dominates \((c - s)q\).)

**Lemma 2.** \((s < c)\). The optimal price and minimax regret for PA2 are

If \( v_{\max} \geq 2v_{\min} - s \), then
\[
p^*(q) = \max_{p \geq v_{\min}} \left( \frac{m_{\max}}{m_{\max} + q} \right) + s \left( \frac{1 - \frac{m_{\max}}{m_{\max} + q}}{m_{\max} - m_{\min}} \right)
\]
\[
\rho^*(p, q) = (v_{\max} - s) \left( \frac{m_{\max}}{m_{\max} + q} \right) m_{\max} - (c - s)q
\]

if \( v_{\max} \leq 2v_{\min} - s \), then
\[
p^*(q) = \begin{cases} 
v_{\max} \left( \frac{m_{\max}}{m_{\max} + q} \right) + s \left( \frac{1 - \frac{m_{\max}}{m_{\max} + q}}{m_{\max} - m_{\min}} \right), & \text{if } q \leq \left(\frac{v_{\max} - v_{\min}}{v_{\max} - s}\right)m_{\max} \\
v_{\min}, & \text{if } q \geq \left(\frac{v_{\max} - v_{\min}}{v_{\max} - s}\right)m_{\max}
\end{cases}
\]
\[
\rho^*(p, q) = \begin{cases} 
(v_{\max} - s) \left( \frac{m_{\max}}{m_{\max} + q} \right) m_{\max} - (c - s)(m_{\max} - q), & \text{if } \frac{v_{\max} - v_{\min}}{v_{\max} - s} m_{\max} \leq q \leq \left( \frac{v_{\max} - v_{\min}}{v_{\max} - s} \right) m_{\max} + \left( \frac{v_{\max} - c}{v_{\max} - s} \right) m_{\max} \\
(v_{\max} - v_{\min}) m_{\min} + (c - s)q - m_{\min}, & \text{if } q \geq \left( \frac{v_{\max} - v_{\min}}{v_{\max} - s} \right) m_{\min} + \left( \frac{v_{\max} - c}{v_{\max} - s} \right) m_{\min}
\end{cases}
\]

**Proof.** Note that
\[
\rho^*(p, q) = \min_p \max_{M \in A, W \in \mathcal{C}} \left\{ \max_{\hat{p} \leq m_{\min}} \left[ E_W[\ell(\hat{p}, \hat{p})] - E_W[\ell(p, q)] \right] \right\} = \min_p \rho^*(p, q) = \min_p \max_p \left\{ g_1(p, q), g_2(p, q) \right\}, \text{ if } p = v_{\min} \\
= \min_p \max_p \left\{ g_3(p, q), g_4(p, q) \right\}, \text{ if } p > v_{\min}
\]
where
\[
g_1(p, q) = (v_{\max} - c)m_{\max} - (p - c)q, g_2(p, q) = (v_{\max} - c)m_{\max} - (p - s)q + (c - s)q, g_3(p, q) = (p - c)m_{\max} + (c - s)q
\]
(see Lemma 1). Note that \( g_1(p, q) \) is decreasing in \( p \) and \( g_2(p, q) \) is increasing in \( p \). Note also that \( g_1(p, q) = g_2(p, q) \) at \( p' = v_{\max} m_{\max} / (m_{\max} + q) + s(1 - (m_{\max} / m_{\max} + q)) \) and that \( p' > v_{\min} \) if \( q < (v_{\max} - v_{\min} / v_{\min} - s)m_{\max} \). Therefore, if \( q \leq (v_{\max} - v_{\min} / v_{\min} - s)m_{\max} \), the optimal price is
\[
p^*(q) = p' = v_{\max} \left( \frac{m_{\max}}{m_{\max} + q} \right) + s \left( \frac{1 - \frac{m_{\max}}{m_{\max} + q}}{m_{\max} - m_{\min}} \right)
\]
\[
\rho^*(q) = \max \left\{ g_3(p^*(q), q), g_4(p^*(q), q) \right\} = g_1(p^*(q), q) = g_2(p^*(q), q) = (v_{\max} - s) \left( \frac{m_{\max}}{m_{\max} + q} \right) m_{\max} - (c - s)(m_{\max} - q)
\]
Note that if \( v_{\max} \geq 2v_{\min} - s \), then \((v_{\max} - v_{\min} / v_{\min} - s)m_{\max} \geq m_{\max} \), and since \( q \leq m_{\max} \), we have \( q \leq (v_{\max} - v_{\min} / v_{\min} - s)m_{\max} \) for any feasible \( q \), and
\[
p^*(q) = p' = v_{\max} \left( \frac{m_{\max}}{m_{\max} + q} \right) + s \left( \frac{1 - \frac{m_{\max}}{m_{\max} + q}}{m_{\max} - m_{\min}} \right)
\]
\[ \rho^*(q) = g_1(p^*(q), q) = (v_{\text{max}} - s) \left( \frac{m_{\text{max}}}{m_{\text{max}} + q} \right) m_{\text{max}} - (c - s)(m_{\text{max}} - q). \]

Alternatively, if \((v_{\text{max}} - v_{\text{min}}/v_{\text{min}} - s)m_{\text{max}} \leq q \leq m_{\text{max}}\), then the optimal price is \(p^*(q) = v_{\text{min}}\) and

\[ \rho^*(q) = \max \{ g_1(v_{\text{min}}, q), g_2(v_{\text{min}}, q) \} = \begin{cases} 
    g_1(v_{\text{min}}, q), & \text{if } q \leq \min \{m_{\text{min}}, q\} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)(m_{\text{max}} - m_{\text{min}}) \\
    g_2(v_{\text{min}}, q), & \text{if } q \geq \min \{m_{\text{min}}, q\} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)(m_{\text{max}} - m_{\text{min}}).
\end{cases} \]

The above expression can expressed more precisely by considering the case of \(q \leq m_{\text{min}}\) and \(q > m_{\text{min}}\). If \(q \leq m_{\text{min}}\), then \(\min \{m_{\text{min}}, q\} = q\) and \(\min \{m_{\text{min}}, q\} + \left( v_{\text{max}} - c/v_{\text{min}} - s \right) (m_{\text{max}} - m_{\text{min}}) = q + \left( v_{\text{max}} - c/v_{\text{min}} - s \right) (m_{\text{max}} - m_{\text{min}}) \geq q\). Thus, if \((v_{\text{max}} - v_{\text{min}}/v_{\text{min}} - s)m_{\text{max}} \leq q\) and \(q \leq m_{\text{min}}\), then

\[ \rho^*(q) = g_1(v_{\text{min}}, q). \]

If \((v_{\text{max}} - v_{\text{min}}/v_{\text{min}} - s)m_{\text{max}} \leq q \leq m_{\text{max}}\) and \(q > m_{\text{min}}\), then \(\min \{m_{\text{min}}, q\} = m_{\text{min}}\) and we have

\[ m_{\text{min}} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)(m_{\text{max}} - m_{\text{min}}) = \left( 1 - \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right) m_{\text{min}} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right) m_{\text{max}} = \left( \frac{v_{\text{max}} - v_{\text{min}}}{v_{\text{min}} - s} \right) m_{\text{max}} \]

The inequality follows from our assumption that \(v_{\text{min}} - c \geq 0\) (e.g., consumers will purchase the product if the retailer sells at cost) and the inequality \((v_{\text{max}} - v_{\text{min}}/v_{\text{min}} - s) \leq 1\) that is a consequence of \((v_{\text{max}} - v_{\text{min}}/v_{\text{min}} - s)m_{\text{max}} \leq m_{\text{max}}\). Also, note that \((1 - (v_{\text{max}} - c/v_{\text{min}} - s)m_{\text{max}} + (v_{\text{max}} - c/v_{\text{min}} - s)m_{\text{max}} > m_{\text{max}}\) (due to \(m_{\text{min}} < m_{\text{max}}\)); this assures that when \(q \geq \min \{m_{\text{min}}, q\} + (v_{\text{max}} - c/v_{\text{min}} - s)m_{\text{max}} - m_{\text{min}}\), we also have \(q > m_{\text{min}}\). Thus, if \((v_{\text{max}} - v_{\text{min}}/v_{\text{min}} - s)m_{\text{max}} \leq q \leq m_{\text{max}}\) and \(q > m_{\text{min}}\), then

\[ \rho^*(q) = \begin{cases} 
    g_1(v_{\text{min}}, q), & \text{if } q \leq \left( 1 - \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{min}} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{max}} \\
    g_2(v_{\text{min}}, q), & \text{if } q \geq \left( 1 - \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{min}} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{max}} \end{cases} \]

In summary, if \(v_{\text{max}} \geq 2v_{\text{min}} - s\), then

\[ \rho^*(q) = \begin{cases} 
    \max \{m_{\text{max}}/m_{\text{max}} + q, s(1 - m_{\text{max}}/m_{\text{max}} + q)\}, & \text{if } q \leq (v_{\text{min}} - m_{\text{max}}) m_{\text{max}} \\
    v_{\text{min}}, & \text{if } q \geq \frac{(v_{\text{min}} - m_{\text{max}}) m_{\text{max}}}{v_{\text{min}} - s} m_{\text{max}} \end{cases} \]

\[ \rho^*(q) = \begin{cases} 
    g_1(p^*(q), q), & \text{if } q \leq \left( 1 - \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{min}} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{max}} \\
    g_2(p^*(q), q), & \text{if } q \geq \left( 1 - \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{min}} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{max}} \end{cases} \]

\[ = \begin{cases} 
    (v_{\text{max}} - c)m_{\text{max}} - (v_{\text{min}} - c)q, & \text{if } q \leq \left( 1 - \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{min}} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{max}} \\
    (v_{\text{max}} - v_{\text{min}})m_{\text{min}} + (c - s)(q - m_{\text{min}}), & \text{if } q \geq \left( 1 - \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{min}} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right)m_{\text{max}} \end{cases} \]

where

\[ g_1(p, q) = (v_{\text{max}} - c)m_{\text{max}} - (p - c)q \\
g_2(p, q) = (v_{\text{max}} - c)m_{\text{min}} - (p - s)\min \{m_{\text{min}}, q\} + (c - s)q. \]

\[ \Box \]

**Proof of Proposition 1.** The problem is

\[ \rho^* = \min \rho^* = \max_{p^* \leq m_{\text{max}} \in A} \left\{ \max_{\theta \in \mathcal{A}} \left[ E_W[H(\hat{\theta}, \hat{\theta})] - E_W[H(p, q)] \right] \right\} \]

\[ = \min_{q \leq m_{\text{max}}} \rho^*(q) \]

where the functional form of \(\rho^*(q)\) depends on parameter values. In particular, if \(v_{\text{max}} \geq 2v_{\text{min}} - s\), then

\[ \rho^* = \min_{q \leq m_{\text{max}}} g_1(q) \]
The function that is minimized at one of three possible values, the optimal price as a function of \( q \) is

\[
\rho^*(q) = \begin{cases} 
\min_{q \leq m_{\text{max}}} 
& \text{if } q \leq \left( \frac{v_{\text{max}} - \rho_{\text{max}}}{\rho_{\text{min}} - s} \right) m_{\text{max}} \\
\max_{q \geq m_{\text{min}}} 
& \text{if } q \geq \left( \frac{v_{\text{max}} - v_{\text{min}}}{v_{\text{min}} - s} \right) m_{\text{max}} \\
\frac{m_{\text{max}} + q}{\rho_{\text{min}} - s} + \left( \frac{m_{\text{max}} - q}{\rho_{\text{max}} - s} \right) m_{\text{max}} 
& \text{if } q \in \left( \frac{v_{\text{max}} - \rho_{\text{min}}}{v_{\text{min}} - s}, \frac{v_{\text{max}} - \rho_{\text{max}}}{\rho_{\text{max}} - s} \right) m_{\text{max}} 
\end{cases}
\]

where the optimal price as a function of \( q \) is

\[
\rho^*(q) = \begin{cases} 
\frac{m_{\text{max}} + q}{\rho_{\text{min}} - s} + \left( \frac{m_{\text{max}} - q}{\rho_{\text{max}} - s} \right) m_{\text{max}} 
& \text{if } q \in \left( \frac{v_{\text{max}} - \rho_{\text{min}}}{v_{\text{min}} - s}, \frac{v_{\text{max}} - \rho_{\text{max}}}{\rho_{\text{max}} - s} \right) m_{\text{max}} 
\end{cases}
\]

and

\[
g_1(q) = (v_{\text{max}} - s)m_{\text{max}}^2 / (m_{\text{max}} + q) - (c-s)(m_{\text{max}} - q)g_2(q) = (v_{\text{max}} - c)m_{\text{max}} - (v_{\text{min}} - c)g_3(q) = (v_{\text{max}} - v_{\text{min}})m_{\text{min}} + (c-s)(q - m_{\text{min}})
\]

(see Lemma 2). The function \( g_1(q) \) is convex, \( g_2(q) \) is linear decreasing, and \( g_3(q) \) is linear increasing. Thus, \( \rho^*(q) \) is a piecewise convex function that is minimized at one of three possible values, the first two of which can be written as functions of \( v_{\text{max}} \) (see Fig. A1):

1. The stationary point of \( g_1(q) \),
   \[
   q_1(v_{\text{max}}) = \left( \frac{v_{\text{max}} - s}{c - s} \right)^{1/2} - 1 \right) m_{\text{max}}.
   \]

2. The point where \( g_2(q) \) and \( g_3(q) \) intersect
   \[
   q_3(v_{\text{max}}) = \left( 1 - \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right) m_{\text{min}} + \left( \frac{v_{\text{max}} - c}{v_{\text{min}} - s} \right) m_{\text{max}}.
   \]

3. The upper limit on a viable order quantity: \( m_{\text{max}} \)

What remains is to characterize conditions under which each of the above quantities is optimal. We define the larger of the two quantities where \( g_1(q) \) and \( g_2(q) \) intersect as

\[
q_2(v_{\text{max}}) = \left( \frac{v_{\text{max}} - v_{\text{min}}}{v_{\text{min}} - s} \right) m_{\text{max}}
\]

(\( g_1(q) \) and \( g_2(q) \) also intersect at \( q = 0 \)). Let

\[
\beta = m_{\text{min}} / m_{\text{max}}
\]

\[
g_4(q) = \begin{cases} 
g_2(q), & \text{if } q \leq q_3 \\
g_3(q), & \text{if } q \geq q_3
\end{cases}
\]

\[
q_4 = \arg \min_{q \leq m_{\text{max}}} g_4(q) = \min \{ m_{\text{max}}, q_3 \}
\]

\[
\begin{align*}
v_1 &= 4c - 3s \\
v_2 &= 2v_{\text{min}} - 5 \\
v_3 &= v_{\text{min}} + 2(v_{\text{min}} - c)(c - s)^{1/2} \\
v_4 &= s + (v_{\text{min}} - s)^2 / (c-s)
\end{align*}
\]

Fig. A1. Illustration of \( \rho^*(q) \) with parameters \( s = 0, c = 10, v_{\text{min}} = 15, v_{\text{max}} = 20, m_{\text{min}} = 0, m_{\text{max}} = Q = 100 \). The stationary point of \( g_1(q) \) is \( q_1 = 41.1 \), curves \( g_1(q) \) and \( g_2(q) \) intersect at \( q_2 = 33.3 \), curves \( g_3(q) \) and \( g_4(q) \) intersect at \( q_3 = 66.7 \), and the minimax regret order quantity is \( \tilde{q} = 66.7 \).
\[ v_5 = v_{\text{min}} + c - s \]
\[ v_6 = s + (c-s) \left( \frac{1 + (1-2a+ab)^{1/2}}{a} \right)^2 \]

where
\[ a = \frac{m_{\text{min}}}{m_{\text{max}}} + \left( \frac{1}{m_{\text{max}}} \right) \left( \frac{c-s}{v_{\text{min}}-s} \right) \]
\[ b = a \cdot \frac{m_{\text{min}}}{m_{\text{max}}} \left( \frac{v_{\text{min}}-c}{c-s} \right) = \left( 1 - \frac{m_{\text{min}}}{m_{\text{max}}} \right) \left( \frac{c-s}{v_{\text{min}}-s} \right) + \frac{m_{\text{min}}}{m_{\text{max}}} \left( \frac{v_{\text{min}}-c}{c-s} \right) \]

The following relationships can be obtained (e.g., via algebraic manipulation):

1. \( q_3(v_{\text{max}}) \leq m_{\text{max}} \) iff \( v_{\text{max}} \leq v_1 \) (with equality at \( v_{\text{max}} = v_1 \))
2. \( q_2(v_{\text{max}}) \leq m_{\text{max}} \) iff \( v_{\text{max}} \leq v_2 \) (with equality at \( v_{\text{max}} = v_2 \))
3. \( q_3(v_{\text{max}}) \leq m_{\text{max}} \) iff \( v_{\text{max}} \leq v_3 \) (with equality at \( v_{\text{max}} = v_3 \))
4. \( q_1(v_{\text{max}}) \leq q_2(v_{\text{max}}) \) iff \( v_{\text{max}} \geq v_4 \) (with equality at \( v_{\text{max}} = v_4 \))
5. \( g_1(q_1) \leq g_2(q_2) \) iff \( v_{\text{max}} \geq v_3 \) (with equality at \( v_{\text{max}} = v_3 \))
6. \( g_1(q_1) \leq g_2(q_2) \) iff \( v_{\text{max}} \geq v_4 \) (with equality at \( v_{\text{max}} = v_4 \))
7. \( v_1 > v_2 > v_3 > v_4 \) if \( v_{\text{min}} = c \)
8. \( v_1 > v_2 > v_4 > v_3 \) if \( v_{\text{min}} = c + (c-s) \)
9. \( v_1 = v_2 = v_3 = v_4 \) if \( v_{\text{min}} = c + (c-s) \)
10. \( v_1 < v_2 < v_4 < v_3 \) if \( v_{\text{min}} > c + (c-s) \)
11. \( v_2 \leq v_4 \) if \( v_{\text{min}} \leq c + (c-s)/4 \) (with equality at \( v_{\text{min}} = c + (c-s)/4 \))
12. \( v_5 \leq v_3 \) if \( v_{\text{min}} \leq c + (c-s)/4 \) (with equality at \( v_{\text{min}} = c + (c-s)/4 \))
13. \( v_4 \leq v_6 \) if \( v_{\text{min}} \leq c + (c-s)/4 \)

We consider two cases.

**Case 1.** \( v_{\text{max}} \geq 2v_{\text{min}} - s \) (\( = v_2 \)). For this case, the problem is
\[ \rho^* = \min_{q \leq m_{\text{max}}} g_1(q), \]
and the optimal quantity is clearly
\[ q^* = \min \{ q_1, m_{\text{max}} \}. \]

Using inequalities (1)–(12) we can express the optimal quantity more precisely
\[ q^* = q_1 \quad \text{if} \quad v_{\text{max}} \leq v_1 \]
\[ q^* = m_{\text{max}} \quad \text{if} \quad v_{\text{max}} \geq v_1 \]

**Case 2.** \( v_{\text{max}} \leq 2v_{\text{min}} - s \) (\( = v_2 \)). For this case, the problem is
\[ \rho^* = \min_{q \leq m_{\text{max}}} \begin{cases} g_1(q) \quad \text{if} \quad q \leq \frac{v_{\text{max}} - v_{\text{min}}}{v_{\text{min}} - s} \frac{m_{\text{max}}}{m_{\text{min}}} \\ g_2(q) \quad \text{if} \quad \frac{v_{\text{max}} - v_{\text{min}}}{v_{\text{min}} - s} \frac{m_{\text{max}}}{m_{\text{min}}} \leq q \leq \left( 1 - \frac{m_{\text{min}}}{m_{\text{max}}} \right) \frac{m_{\text{min}}}{m_{\text{max}}} + \frac{m_{\text{min}}}{m_{\text{max}}} \frac{v_{\text{min}} - c}{c-s} \\ g_3(q) \quad \text{if} \quad q \geq \left( 1 - \frac{m_{\text{min}}}{m_{\text{max}}} \right) \frac{m_{\text{min}}}{m_{\text{max}}} + \frac{m_{\text{min}}}{m_{\text{max}}} \frac{v_{\text{min}} - c}{c-s} \end{cases} \]

and, noting that \( q_2 \leq q_3 \) (due to \( v_{\text{max}} \leq v_2 \)), the optimal quantity is clearly

- \( q^* = q_4 \quad \text{if} \quad q_1 \geq m_{\text{max}} \)
- \( q^* = q_4 \quad \text{if} \quad q_1 \leq q_2 \)
- \( q^* = q_4 \quad \text{if} \quad q_1 \leq m_{\text{max}}, q_1 \leq q_2, \) and \( g_1(q_1) \leq g_2(q_2) \)
- \( q^* = q_4 \quad \text{if} \quad q_1 \leq m_{\text{max}}, q_1 \leq q_2, \) and \( g_1(q_1) \geq g_2(q_2) \)

Using inequalities (1)–(13) we can express the optimal quantity more precisely.

If \( v_{\text{min}} \geq c - s \), then
\[ q^* = q_4 = \min \{ m_{\text{max}}, q_3 \} = \begin{cases} q_3 \quad \text{if} \quad v_{\text{max}} \leq v_5 \\ m_{\text{max}} \quad \text{if} \quad v_{\text{max}} \geq v_5 \end{cases} \]

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3 This result is obtained through logic, rather than algebraic manipulation: At \( v_{\text{max}} = c + (c-s)/4 \), we have \( v_5 = v_4 \) (due to (11)). Observe that at \( v_{\text{max}} = v_5 \), we have \( q_3 = m_{\text{max}} \) (due to (3), and thus \( g_1(q_1) = g_2(m_{\text{max}}) \) implies \( v_{\text{max}} = v_5 \) (due to (6)), and thus \( v_5 = v_4 \) when \( v_{\text{min}} = c + (c-s)/4 \) (and for any value of \( v_{\text{max}} \)). As an aside, if \( v_{\text{min}} = m_{\text{max}} \), then \( v_5 = v_3 = v_2 \) for all \( v_{\text{min}} \)

4 Setting \( v_4 = v_0 \) and solving for \( v_{\text{min}} \) yields \( v_{\text{min}} = c \) and \( v_{\text{min}} = s + (c-s) \left( 1 + \beta + \frac{1}{2} (1-\beta)^2 \right) \left( 1 - \beta \right) / 2 \geq c + (c-s)/4 \) for all \( \beta \in [0,1] \). The direction of the inequality follows from comparing \( v_4 \) with \( v_5 \) for any \( v_{\text{min}} \in (c, c+(c-s)/4] \).

5 Note that \( q_3 = q_2 + \left[ 1 - \frac{1}{2} \left( \frac{v_{\text{max}} - v_{\text{min}}}{v_{\text{min}} - s} \right) \frac{m_{\text{max}}}{m_{\text{min}}} \right] m_{\text{min}} + \frac{(v_{\text{min}} - c)(v_{\text{min}} - s)(m_{\text{min}} - m_{\text{max}})}{(v_{\text{min}} - s)(m_{\text{min}} - m_{\text{max}})} \). Furthermore, \( v_{\text{max}} \leq v_2 \) implies \( (v_{\text{max}} - v_{\text{min}})(v_{\text{min}} - s) \leq 1 \), and since \( m_{\text{max}} \geq m_{\text{min}} \) and \( v_{\text{min}} \geq c \), it follows that \( q_3 \geq q_2 \).
This is because \( v_1 \leq v_2 \leq v_4 \) and

- for \( v_{\text{max}} \leq v_1 \leq v_4 \): \( q_1 \geq q_2 \Rightarrow q^* = \min \{ m_{\text{max}}, q_3 \} = \begin{cases} q_4, & \text{if } v_{\text{max}} \leq v_5 \\ m_{\text{max}}, & \text{if } v_{\text{max}} \geq v_5 \end{cases} \)
  - for \( v_{\text{max}} \geq v_1 \): \( q_1 \geq m_{\text{max}} \Rightarrow q^* = q_4 = \min \{ m_{\text{max}}, q_3 \} = \begin{cases} q_4, & \text{if } v_{\text{max}} \leq v_5 \\ m_{\text{max}}, & \text{if } v_{\text{max}} \geq v_5 \end{cases} \).

If \( (c-s)/4 \leq v_{\text{min}} - c \leq c - s \), then

\[
q^* = q_4 = \min \{ m_{\text{max}}, q_3 \} = \begin{cases} q_4, & \text{if } v_{\text{max}} \leq v_5 \\ m_{\text{max}}, & \text{if } v_{\text{max}} \geq v_5 \end{cases},
\]

\[
q^* = q_1 \text{ if } v_3 \leq v_{\text{max}} \leq v_2.
\]

This is because, for \( v_{\text{min}} - c \in [(c-s)/4, c-s] \), we have \( v_1 \geq v_2 \geq v_3 \geq v_4 \) and \( v_6 \geq v_5 \geq v_4 \) and

- for \( v_1 \geq v_2 \geq v_{\text{max}} \geq v_3 \), which implies \( v_{\text{max}} \geq v_4 \) and \( v_{\text{max}} \geq v_5 \): \( q_1 \leq m_{\text{max}}, g_1(q_1) \leq g_2(m_{\text{max}}), q_1 \leq q_2, q_3 \geq m_{\text{max}} \Rightarrow q_4 = m_{\text{max}}, g_1(q_1) \leq g_4(q_4) = g_4(m_{\text{max}}), \) and \( q^* = q_1 \)
  - for \( v_1 \geq v_2 \geq v_3 \geq v_{\text{max}} \) which implies \( v_{\text{max}} \leq v_6 \): \( q_1 \leq m_{\text{max}}, g_2(m_{\text{max}}) \leq g_1(q_1), g_3(q_3) \leq g_2(q_4) \Rightarrow g_4(q_4) \leq g_1(q_1) \leq g_1(q_3) \) and \( q^* = q_4 \).

If \( v_{\text{min}} - c \leq (c-s)/4 \), then

\[
q^* = q_4 \text{ if } v_{\text{max}} \leq v_6
\]

\[
q^* = q_1 \text{ if } v_5 \leq v_{\text{max}} \leq v_2.
\]

This is because \( v_4 \leq v_3 \leq v_2 \leq v_1 \) and \( v_6 \leq v_3 \leq v_5 \) and \( v_4 \leq v_6 \) and

- for \( v_1 \geq v_2 \geq v_{\text{max}} \geq v_6 \geq v_4 \) and \( v_{\text{max}} \leq v_5 \): \( q_1 \leq m_{\text{max}}, q_2 \leq m_{\text{max}}, g_1(q_1) \leq g_2(q_2), q_1 \leq m_{\text{max}}, q_2 \leq q_3, g_1(q_1) \leq g_4(q_4) \), and \( q^* = q_1 \)
  - for \( v_1 \geq v_2 \geq v_{\text{max}} \geq v_4 \) and \( v_{\text{max}} \geq v_5 \): \( q_1 \leq m_{\text{max}}, q_2 \leq m_{\text{max}}, g_1(q_1) \leq g_2(q_2), q_3 \geq m_{\text{max}}, q_1 \leq q_2 \Rightarrow q_4 = m_{\text{max}}, g_4(q_4) = g_2(m_{\text{max}}) \)
  - for \( v_1 \geq v_2 \geq v_3 \geq v_{\text{max}} \) which implies \( v_{\text{max}} \leq v_5 \): \( q_1 \leq m_{\text{max}}, q_2 \leq m_{\text{max}}, g_2(q_2) \leq g_3(q_3), q_3 \leq m_{\text{max}} \Rightarrow q_4 = q_3, g_4(q_4) = g_2(q_3) \leq g_1(q_1) \) and \( q^* = q_3 \).

Summarizing the two cases, we have

If \( v_{\text{min}} - c \leq c - s \), then

\[
q^* = q_3 \text{ if } v_{\text{max}} \leq v_5
\]

\[
q^* = m_{\text{max}} \text{ if } v_{\text{max}} \geq v_5
\]

If \( (c-s)/4 \leq v_{\text{min}} - c \leq c - s \), then

\[
q^* = q_1 \text{ if } v_{\text{max}} \leq v_5
\]

\[
q^* = m_{\text{max}} \text{ if } v_5 \leq v_{\text{max}} \leq v_3
\]

\[
q^* = q_1 \text{ if } v_5 \leq v_{\text{max}} \leq v_1
\]

\[
q^* = m_{\text{max}} \text{ if } v_{\text{max}} \geq v_1.
\]

The expressions for optimal price and minimax regret are obtained by substituting \( q^* \) into \( p^*(q) \) and \( p^*(q) \).

References


