3 Dimensional Isotropic Tensors

If we can reduce possibilities by considering special cases, then certainly the general case possibilities are similarly reduced

for zero rotation $a_{ij} = \delta_{ij}$

consider an infinitesimal rotation $a_{ij} = \delta_{ij} + \beta_{ij}$

where $|\beta_{ij}| << 1$

now

$$a_{ij} a_{ik} = \delta_{jk}$$

subst: $(\delta_{ij} + \beta_{ij})(\delta_{ik} + \beta_{ik}) = \delta_{jk}$

$$\delta_{ij}\delta_{ik} + \delta_{ij}\beta_{ik} + \delta_{ik}\beta_{ij} + \underbrace{\beta_{ij}\beta_{ik}}_{\text{negl}} = \delta_{jk}$$

$$\therefore \beta_{jk} + \beta_{kj} = 0$$

$\therefore \beta$ antisymmetric

[illegible]

$\therefore$ an infinitesimal rotation is of the form $\delta_{ij} + \beta_{ij}$ with β_{ij} antisym.

Look for isotropic 1st order

$$I_k' = C_{mk} I_m \overset{\text{a tensor}}{} = I_k \overset{\text{isotropic}}{}$$

$$I_k = C_{mk} I_m = (\delta_{mk} + \beta_{mk}) I_m$$

$$I_k = I_k + \beta_{mk} I_m$$

$$\beta_{mk} I_m = 0$$

$$\therefore \beta_{1k} I_1 + \beta_{2k} I_2 + \beta_{3k} I_3 = 0$$

or (since β_{ij} antisym)

or $i \to 12, 31, 23$

$$\beta_{21} I_2 + \beta_{31} I_3 = 0 \qquad -\beta_{12} I_2 + \beta_{31} I_3 = 0$$

$$\beta_{12} I_1 + \beta_{32} I_3 = 0 \qquad \beta_{12} I_1 - \beta_{23} I_3 = 0$$

$$\beta_{13} I_1 + \beta_{23} I_2 = 0 \qquad -\beta_{13} + \beta_{23} I_2 = 0$$

consider successive rotations

$$\ni \beta_{21} = 0, \ \beta_{31} \neq 0 \quad \therefore I_3 = 0$$

etc

$\therefore$ only $(0,0,0)$ will do

$\therefore \exists$ no 1st order isotropic tensor under even an inf rotation.

Second Order?

$$I'_{k\ell} = \underset{\substack{\uparrow \\ \text{tensor}}}{C_{rk} C_{s\ell} I_{rs}} = \underset{\substack{\uparrow \\ \text{isotropic}}}{I_{k\ell}}$$

$$I_{k\ell} = (\delta_{rk} + \beta_{rk})(\delta_{s\ell} + \beta_{s\ell}) I_{rs}$$

$$= \Big(\delta_{rk}\delta_{s\ell} + \delta_{rk}\beta_{s\ell} + \delta_{s\ell}\beta_{rk} + \overset{\text{negl}}{\cancel{\beta_{rk}\beta_{s\ell}}}\Big) I_{rs}$$

$$= I_{k\ell} + \underbrace{\beta_{s\ell} I_{ks} + \beta_{rk} I_{r\ell}}_{=0}$$

β antisym

for $k = \ell = 1$: $\beta_{s1} I_{1s} + \beta_{r1} I_{r1} = \cancel{\beta_{11} I_{11}} + \beta_{21} I_{12} + \beta_{31} I_{13} + \cancel{\beta_{11} I_{11}} + \beta_{21} I_{21} + \beta_{31} I_{31}$

$$= -\beta_{12}(I_{12} + I_{21}) + \beta_{31}(I_{31} + I_{13}) = 0$$

for arbitrary β_{ij}

hence $I_{12} + I_{21} = 0$ (1)

$I_{31} + I_{13} = 0$ (2)

similarly for $k = \ell = 2$ $I_{23} + I_{32} = 0$ (3)

$I_{31} + I_{13} = 0$ (2 repeated)

etc only 3 indep

$\kappa = 1,\ l = 2 \qquad \beta_{s2} I_{1s} + \beta_{r1} I_{r2} = \beta_{12} I_{11} + \beta_{32} I_{13} + \beta_{21} I_{22} + \beta_{31} I_{32}$

$$= \beta_{12}(I_{11} - I_{22}) + \beta_{31} I_{32} - \beta_{23} I_{13} = 0$$

for arb. β_{ij}

hence $I_{11} = I_{22} \qquad I_{13} = I_{31} = I_{32} = I_{23} = 0$

& sim $\qquad I_{33} = I_{11} \qquad I_{12} = I_{21} = 0$

$\therefore \quad I_{\kappa l} = \lambda \delta_{\kappa l}$ for an arbitrary infintesimal rot

How about finite trf?

try it out –

take $\quad B_{\kappa l} = \lambda \delta_{\kappa l} \qquad B'_{\kappa l} = \bar{\lambda} \delta_{\kappa l}$

Prove $\quad B'_{\kappa l} = C_{m\kappa} C_{nl} B_{mn}$

$$\bar{\lambda} \delta_{\kappa l} = C_{m\kappa} C_{nl} \lambda \delta_{mn} = \lambda \delta_{\kappa l}$$

OK

Higher order isotropic tensors – 3rd order

$$I'_{mnp} = c_{rm} c_{sn} c_{tp} I_{rst} = I_{mnp}$$

$$(\delta_{rm} + \beta_{rm})(\delta_{sn} + \beta_{sn})(\delta_{tp} + \beta_{tp}) I_{rst} = I_{mnp}$$

$$\left[\delta_{rm}\delta_{sn}\delta_{tp} + \delta_{rm}\delta_{sn}\beta_{tp} + \delta_{rm}\delta_{tp}\beta_{sn} + \delta_{sn}\delta_{tp}\beta_{rm} + (\beta^2 \text{ terms})\right] I_{rst} = I_{mnp}$$

$$I_{mnp} = I_{mnp} + \underbrace{\beta_{tp} I_{mnt} + \beta_{sn} I_{msp} + \beta_{rm} I_{rnp}}_{=0}$$ switch dummies → k

rewrite $\beta_{kp} I_{mnk} + \beta_{kn} I_{mkp} + \beta_{km} I_{knp} = 0$ live subs: mnp

To simplify "bookkeeping" consider 3 cases

a) all subs same $m = n = p = 1$, say

b) 2 " equal $m = n = 1,\ p = 2$, "

c) all " different $m = 1,\ n = 2,\ p = 3$, "

a) $\beta_{k1}(I_{11k} + I_{1k1} + I_{k11}) = 0$

expanding: $[\beta_{11} = 0]$

$$\beta_{21}\underbrace{(I_{112} + I_{121} + I_{211})}_{0} + \beta_{31}\underbrace{(I_{113} + I_{131} + I_{311})}_{0} = 0 \quad \text{for any } \beta_{ij}$$

& same for $m=n=p=2$, $m=n=p=3$

$$I_{112} + I_{121} + I_{211} = 0$$

$$I_{113} + I_{131} + I_{311} = 0$$

$$I_{223} + I_{232} + I_{322} = 0$$

$$I_{221} + I_{212} + I_{122} = 0$$

$$I_{331} + I_{313} + I_{133} = 0$$

$$I_{332} + I_{323} + I_{233} = 0$$

(b) $\beta_{k2} I_{11k} + \beta_{k1}(I_{1k2} + I_{k12}) = 0$

$\beta_{12} I_{111} + \beta_{32} I_{113} + \beta_{21}(I_{122} + I_{212}) + \beta_{31}(I_{132} + I_{312}) = 0$

$I_{113} = 0$ (1) $\quad I_{122} + I_{212} = I_{111}$ (2) $\quad I_{132} + I_{312} = 0$ (3)

if exactly 2 subs equal the element is zero $\longrightarrow$ if all 3 subs equal element is zero

$I_{132} = -I_{312}$

cyclic permutations $\quad I_{213} = -I_{123}$ --OK

$I_{321} = -I_{231}$

$m = n = 1,\ p = 3$

$\beta_{k3} I_{11k} + \beta_{k1}(I_{1k3} + I_{k13}) = 0$

only non zero term is $\beta_{21}(I_{123} + I_{213})$

$I_{123} = -I_{213}$

$I_{231} = -I_{321}$

$I_{312} = -I_{132}$

$I_{123} = I_{312} = I_{231} = -I_{321} = -I_{132} = -I_{213} = +\lambda$

alt tensor

Recognizing that the 3-D, 3rd order R.C. isotropic tensor

I_{ijk} satisfies the conditions $= 1$ if ijk is an even permutation of 123

-1 " " odd " "

0 " " " no " "

we call this the alternating tensor in 3-D

Some properties of alternating tensor

$$\left.\begin{aligned} \epsilon_{pqr}\epsilon_{pqr} &= 6 \\ \epsilon_{pqi}\epsilon_{pqj} &= 2\delta_{ij} \\ \epsilon_{pij}\epsilon_{pkl} &= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \end{aligned}\right\} \underline{\text{exercises}}$$

Associates a vector : DUAL VECTOR
with a tensor :

$$t_i = \epsilon_{ijk} T_{jk}$$

note: depends only on antisymmetric part of T_{jk}

inverse if anti-sym: