The Fundamental Integral Theorem in 3-D Continuum

If g and its first derivatives are contin in a volumetric region V bounded by a closed surface S then:

$$\int_V g_{,i} = \int_S g n_i \qquad ①$$

note g may be shortened form of $g_{ikl\cdots,pqr\cdots}$

Restrict the region to one that is "x_i simple"
ie any line $\parallel x_i$ axis "pierces" S no more than twice
(note prismatic type OK —
where part of S is $\parallel$ to x_i axis)

① $\Rightarrow$

$$\int_V g_{,1} = \int_S g n_1 \qquad ②$$

$$\int_V g_{,2} = \int_S g n_2 \qquad ③$$

$$\int_V \frac{\partial g}{\partial x_3}\, dv = \int_S g n_3\, ds \qquad ④$$

Since there is no "preferred" axis in either the equations or the restrictions

$\therefore$ proving ④ $\Rightarrow$ ① valid

Consider the surface S in 3 parts

$S^{(1)}$: X_3 component of outward normal is neg ie $n_3 < 0$

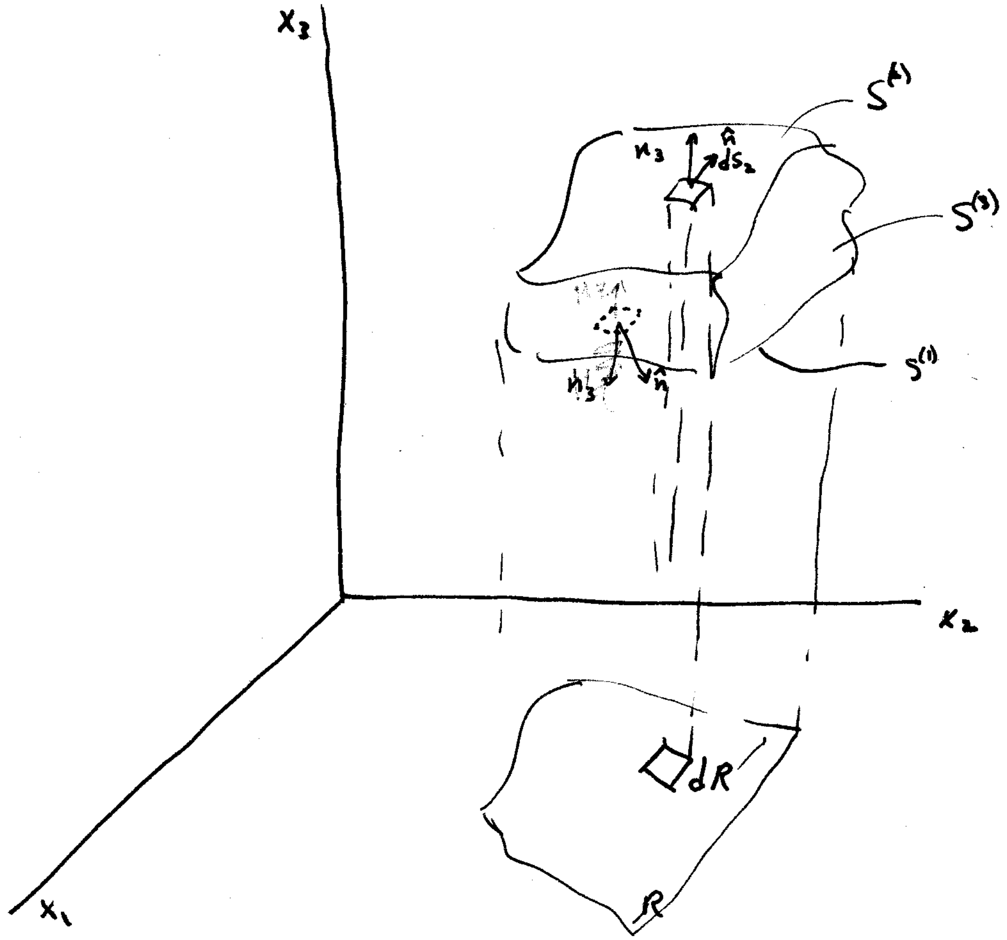
$S^{(2)}$: " pos $n_3 > 0$

$S^{(3)}$: " 0 $n_3 = 0$

let $X_3 = U(x_1, x_2)$ on $S^{(2)}$

$X_3 = V(x_1, x_2)$ on $S^{(1)}$

R is the projection in $X_1 X_2$ plane of S


x_3
$S^{(2)}$
n_3
$\hat{n}$
dS_2
$S^{(3)}$
$S^{(1)}$
n_3
$\hat{n}$
x_2
dR
R
x_1

Writing LS of ④ as iterated integral

$$\int_V g_{,3}\, dV =$$

$$= \iiint \frac{\partial g}{\partial x_3}\, dx_3\, dx_1\, dx_2 = \iint \left[g(S_3^{(2)}) - g(S_3^{(1)}) \right] dx_1\, dx_2$$

$$= \int_R \left[g(S^{(2)}) - g(S^{(1)}) \right] dR$$

looking @ R.S. of 4

$n_3 = 0$ on $S^{(3)}$

$$\int_S g n_3 = \int_{S^{(1)}} g n_3\, dS^{(1)} + \int_{S^{(2)}} g n_3\, dS^{(2)} + \cancel{\int_{S^{(3)}} g n_3\, dS^{(3)}}$$

but we can write $n_3\, dS^{(2)} = dR$

$$-n_3\, dS^{(1)} = dR$$

$$\therefore \int_S g n_3 = \int_R g(S^{(2)})\, dR - \int_R g(S^{(1)})\, dR$$

qed

since RS = LS, thm is proven

this can be extended: eg hourglass figure

Generalizing to 3-D

$$\int_V g_{,k}\, dV = \int_S g\, n_k\, dS$$

special cases (Greens, Gauss, Divergence, Ostrogradskii's thm)

$$\int_V V_{k,k} = \int_S V_k n_k$$

or $$\int_V \nabla \cdot \vec{V} = \int_S V_n$$

$$\int_V \varphi_{,kk} = \int_S \varphi_{,k} n_k$$

or $\int \nabla\varphi \cdot \hat{n}$

or $$\int_V \nabla^2 \varphi = \int_S \frac{d\varphi}{dn}$$