Intro to Continuum Plasticity

Uniaxial tension

nominal stress $\hat{\sigma} = P/A_0$

" strain $\hat{\epsilon} = \dfrac{l - l_0}{l_0}$

true stress $\sigma = P/A$

(log) " strain $\epsilon = \dfrac{dl}{l} \rightarrow \epsilon = \int_0^l \dfrac{dl}{l} = \ln\left(\dfrac{l}{l_0}\right)$

$$\underline{\epsilon} = \ln\left(\frac{l}{l_0}\right) = \ln\left(\frac{l-l_0}{l_0} + \frac{l_0}{l_0}\right) = \underline{\ln\left(1+\hat{\epsilon}\right)}$$

or $\underline{\hat{\epsilon} = e^{\epsilon} - 1}$

If (As in Experimental): $\underline{\text{incompressible}}$ · $A_0 l_0 = A l$
($\nu = .5$)

$$\underline{\sigma} = \frac{P}{A} = \frac{P}{A_0}\frac{l}{l_0} = \underline{\hat{\sigma}\left(1+\hat{\epsilon}\right)} = \hat{\sigma} e^{\epsilon}$$

or $\underline{\hat{\sigma} = \sigma e^{-\epsilon}}$

Bridgeman expts

1. Incompr.
2. stress/strain indep. of hydrostatic pressure
3. Ductility increases under hyd. pr.
4. Yield indep of " "

Incomp → $h w d = h_o w_o d_o$

$$h_o(1+\hat{\epsilon}_h)\, w_o(1+\hat{\epsilon}_w)\, d_o(1+\hat{\epsilon}_d) = h_o w_o d_o$$

or $(1+\hat{\epsilon}_h)(\quad)(\quad) = 1$

or $e^{\epsilon_h}\, e^{\epsilon_w}\, e^{\epsilon_d} = 1$

$\ln(\quad " \quad) = 0$

$$\epsilon_h + \epsilon_w + \epsilon_d = 0$$

recall $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

thus

$$e^{\epsilon} = 1 + \epsilon + \frac{\epsilon^2}{2!} + \cdots$$

$$\therefore \quad \hat{\epsilon} = e^{\epsilon} - 1 = 1 + \epsilon + \frac{\epsilon^2}{2!} + \cdots - 1$$

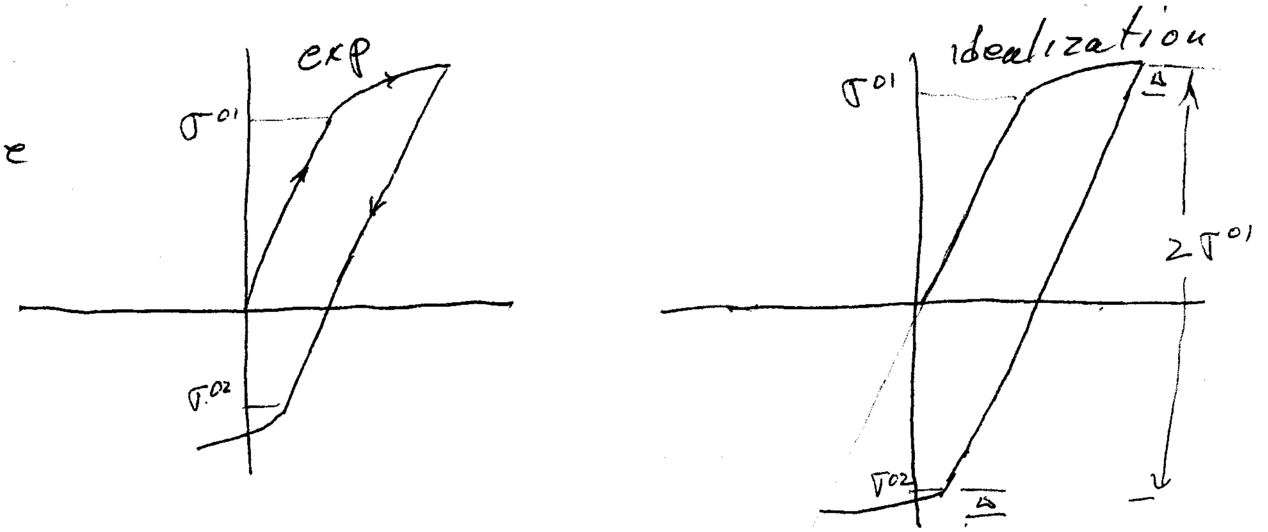
or $\quad \hat{\epsilon} \approx \epsilon \qquad$ for $\epsilon << 1$

or $\quad \hat{\epsilon}_h + \hat{\epsilon}_w + \hat{\epsilon}_d \approx 0$

Bauschinger

Loading in $\left\{\begin{matrix}\text{tension}\\ \text{comp}\end{matrix}\right\}$ into "strain hardening" region causes decrease in yield point in opp. dirn.

ie



Temperature and rate effect

Temp can be important in rocket engines etc

Rate effect " " biological materials

Won't consider them in this course.

Indicial notation

Latin : 1, 2, 3 Greek - anything

In an expression:

Index appearing once : live or free index

" " twice : dummy or summation index

" " 3 or more : MISTAKE!

e.g. $V_i Q_{jk} M_{ij}$ $\quad$ i, j dummies

$= V_1 Q_{1k} M_{11} + V_2 Q_{1k} M_{21} + \cdots$ $\quad$ k live

(9 terms)

$$v_{ii} = v_{11} + v_{22} + v_{33}$$

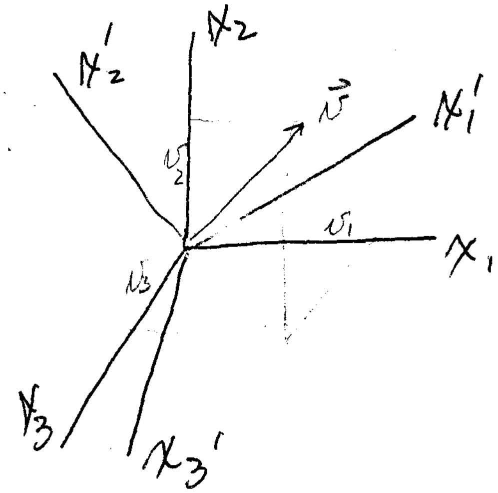
$$b_{iij} = b_{11j} + b_{22j} + b_{33j}$$

Equation

A free index in any term must appear in all

Vectors

vector eg force, vel



express the vector in one (~~unprimed~~ primed)

$$\vec{V} = v_1 \hat{i}_1 + v_2 \hat{i}_2 + v_3 \hat{i}_3$$
$$= v_1' \hat{i}_{1'} + v_2' \hat{i}_{2'} + v_3' \hat{i}_{3'}$$

$$v_1' = v_1 \underbrace{\cos(x_1, x_{1'})}_{a_{11'}} + v_2 \underbrace{\cos(x_2, x_{1'})}_{a_{21'}} + v_3 \underbrace{\cos(x_3, x_{1'})}_{a_{31'}}$$

$$= a_{11'} v_1 + a_{21'} v_2 + a_{31'} v_3$$

$$= a_{i1'} v_i$$

similarly $\quad v_2' = a_{i2'} v_i \qquad v_3' = a_{i3'} v_i$

or $\boxed{v_{j'} = a_{ij'} v_i}$ let this be def'n of vector

$$\hat{\imath}_{k'} = a_{1k'}\hat{\imath}_1 + a_{2k'}\hat{\imath}_2 + a_{3k'}\hat{\imath}_3$$

$$\hat{\imath}_{j'} = a_{1j'}\hat{\imath}_1 + a_{2k'}\hat{\imath}_2 + a_{3k'}\hat{\imath}_3$$

$$\hat{\imath}_{k'}\cdot\hat{\imath}_{j'} = a_{1k'}a_{1j'} + a_{2k'}a_{2j'} + a_{3k'}a_{3j'}$$

$$= a_{ik'}a_{ij'} = \delta_{ii} = \begin{cases} 1 & k=j \\ 0 & k\neq j \end{cases}$$

since $\hat{\imath}_1' \perp \hat{\imath}_2'$ etc

$$\therefore \quad a_{kj'}v_{j'} = a_{kj'}a_{ij'}v_i$$

$$= \delta_{ki}v_i = v_k$$

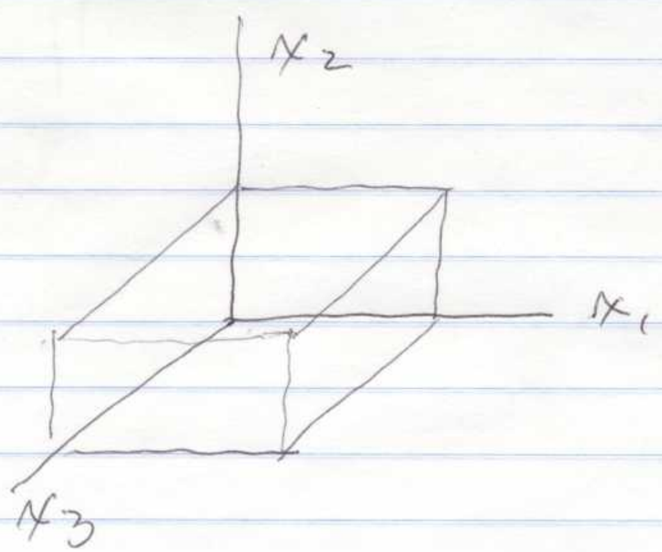
so $\boxed{v_k = a_{kj'}v_{j'}}$ is invers trf

Recall how stress components in one RCS trf to another

$$\sigma_{ij}' = a_{ki}\, a_{lj}\, \sigma_{kl}$$

thus stress is a tensor

Define stress — surface



stress vector

" tensor

Generalized ~~Third order~~ 3-Dimensional Tensors

"G" is an n^{th} order tensor iff

a) it is defined by 3^n nos in any coord system

ie $G_{k_1 k_2 k_3 \dots k_n}$, $G'_{k_1 k_2 \dots k_n}$

and

b)

$$G'_{k_1 k_2 \dots k_n} = a_{m_1 k_1} a_{m_2 k_2} \cdots a_{m_n k_n} G_{m_1 m_2 \dots m_n}$$

or

$$G_{m_1 m_2 \dots m_n} = C_{m_1 k_1} C_{m_2 k_2} \cdots C_{m_n k_n} G'_{k_1 k_2 \dots k_n}$$

Definition Symmetric Tensor $S_{ij} = S_{ji}$

Anti " " $A_{ij} = -A_{ji}$

any tensor $U_{ij} = \frac{1}{2}(U_{ij} + U_{ji}) + \frac{1}{2}(U_{ij} - U_{ji})$

$= S_{ij} + A_{ij}$

proof $(U_{ij} + U_{ji}) = (U_{ji} + U_{ij})$

$(U_{ji} - U_{ij}) = -U_{ij} + U_{ji}$

Show $S_{ij} A_{ij} = 0$ hint switch subs

Given $S_{ij} = S_{ji}$ and S_{ij} a tensor

prove $S'_{ij} = S'_{ji}$

$$\begin{aligned}
S'_{ij} &= a_{mi} a_{nj} S_{mn} \\
&= a_{mi} a_{nj} S_{nm} \\
&= \underbrace{a_{mi} \underbrace{a_{nj} a_{np}}_{\delta_{jp}} a_{mq}}_{\delta_{iq}} S'_{pq} \\
&= S'_{ji}
\end{aligned}$$

Zero Products $\qquad$ scalars: $ab = 0 \rightarrow a$ or $b = 0$

not so for higher order tensors

eg vectors $\quad A_i B_i = 0 \rightarrow A$ or $B = 0$ or $A \perp B$

What if $A_i X_i \equiv 0$ for <u>all</u> X_i

(consider $x_i = (1,0,0)$ etc

$$\Rightarrow A_i \equiv 0$$

applications follow

Given an array of 27 nos defin any coord system
Q_{ijk} in x_i $\qquad$ Q'_{ijk} in x'_i

given also that $Q_{ijk} X_k$ is a second order tensor for all vectors X_k

Then Q_{ijk} is a 3rd order tensor

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define $\quad W_{ij} = Q_{ijk} X_k$
given $\quad W_{ij}$ a 2nd order tensor
X_k a vector
X_k arbitrary

prove $\quad Q_{ijk}$ a 3rd order tensor

$W'_{ij} = a_{mi} a_{nj} W_{mn}$ $\qquad$ W a tensor

$Q'_{ijk} X'_k = a_{mi} a_{nj} Q_{mnp} X_p$

$= a_{mi} a_{nj} Q_{mnp} a_{pk} X'_k$ $\qquad$ X a vector

$\left(Q'_{ijk} - a_{mi} a_{nj} a_{pk} Q_{mnp}\right) X'_k = 0$

for $X_k = (1,0,0)$: $Q'_{ij1} = a_{mi} a_{nj} a_{p1} Q_{mnp}$

$(0,1,0)$ $\quad 2 \quad 2$

$(0,0,1)$ $\quad 3 \quad 3$

$\therefore \quad Q'_{ijk} = a_{mi} a_{nj} a_{pk} Q_{mnp}$

$\therefore$ Q_{ijk} a 3rd order tensor

etc.