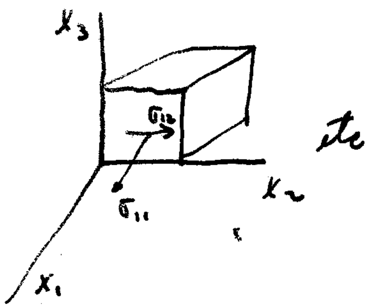
Conservation Theorems

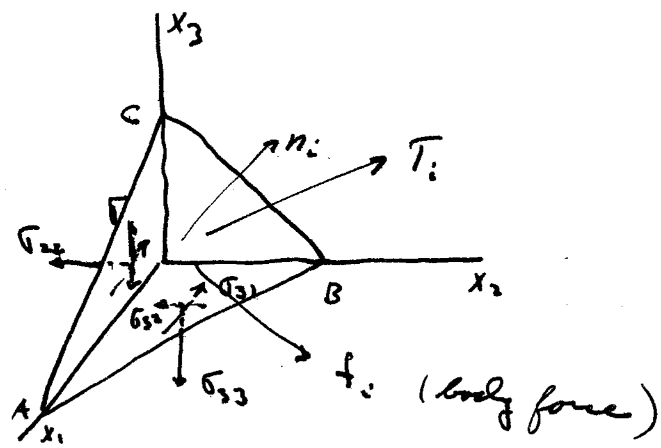
Three-dim stress : consider a cut in a continuum

Stress vector one for each cut -
cut def'd by position + orientation

Stress tensor : associates a stress vector with each direction

or σ_{ij} is the x_j component of the stress vector on plane normal to x_i





area $ABC = \Delta S$ n_i - normal to ΔS

" $AOB = n_3 \Delta S$

$AOC = n_2 \Delta S$ volume : $\frac{1}{3} \Delta S h$

$COB = n_1 \Delta S$

h = altitude from base "ΔS"

$$\int_{ABC} T_i \, dS - \int_{BOC} \sigma_{1i} \, dS - \int_{COA} \sigma_{2i} dS - \int_{AOB} \sigma_{3i} dS + \int_{\Delta V} f_i \, dV = \int_{\Delta V} \frac{d}{dt}(\rho v_i) \, dV$$

using MVT's — mean values denoted by *

$$T_i^* \Delta S - \sigma_{1i}^* n_1 \Delta S - \sigma_{2i}^* n_2 \Delta S - \sigma_{3i}^* n_3 \Delta S + f_i^* \frac{h}{3} \Delta S = \left[\frac{d}{dt}(\rho v_i)\right]^* \frac{h}{3} \Delta S$$

divide then by ΔS + regroup

$$T_i^* - \sigma_{ji}^* n_j + \frac{h}{3} \left\{ f_i^* - \left[\frac{d}{dt}(\rho v_i)\right]^* \right\} = 0$$

now let $h \to 0$

$$T_i = \sigma_{ji} n_j$$

can demonstrate all are tensors

The following occurs so often in CM - let's make sure it is solidly based.

Theorem: If $\int_V g = 0$, $g \in C$, for all regions V

then $g \equiv 0$

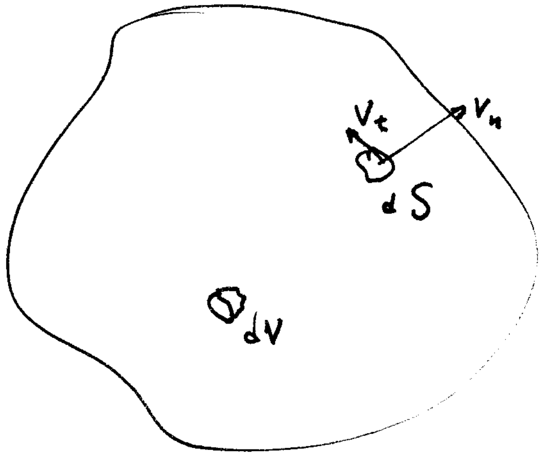
proof: suppose $g \neq 0$ at some point P, say $g > 0$

then, since $g \in C$, $g > 0$ in some sphere V_1 centered at P

then $\int_{V_1} g > 0$ - contradiction

$\therefore$ g cannot be positive

Mass - Control surface (Eulerian)



mass efflux thru dS in time t

$$\rho v_n \, t \, dS \quad \text{(see note)}$$

on entire surface

$$\int_S \rho v_n \, t \, dS = \text{mass efflux from}$$

gain in mass in $dV = \frac{\partial \rho}{\partial t} \, t \, dV$ in V: $\int_V \frac{\partial \rho}{\partial t} \Delta t \, dV$

since the mass efflux - mass increase $= 0$

$$\int_V \frac{\partial \rho}{\partial t} \Delta t \, dV + \int_S \rho v_n \Delta t \, dS = 0$$

we can divide by Δt

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho v_n \, dS = 0$$

{Note : $v_n = v_i n_i$ the subscript n is part of the label} since the normal component of velocity is the scalar product of the velocity + the unit normal

$$v_n = v_i n_i$$

thus

$$\int_S \rho v_n = \int_S \rho v_i n_i = \int_V (\rho v_i)_{,i}$$

$$\therefore \int_V \left[\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} \right] = 0$$

Returning to $\int_V \left[\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} \right] = 0$

this is true for arbitrary V hence

$$\boxed{\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0}$$ continuity equation

steady state : $(\rho v_i)_{,i} = 0$

incompr. : $v_{i,i} = 0$

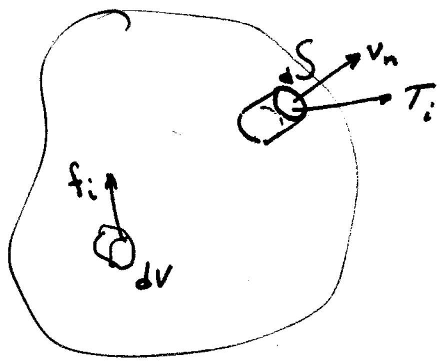
this may be changed in form,

$$(\rho v_i)_{,i} = \rho_{,i} v_i + \rho v_{i,i}$$

thus $\frac{\partial \rho}{\partial t} + v_i \rho_{,i} + \rho v_{i,i} = 0$

we will presently recognize the first two terms as $\frac{D\rho}{Dt}$

Momentum



mass efflux : $\rho v_n \Delta t \, dS$

momentum " : $\rho v_n \Delta t \, dS \, v_i$

total " " : $\int_S \rho v_n \Delta t \, v_i \, dS$

momentum in dV : $\rho \, dV \, v_i$

change " " : $\frac{\partial}{\partial t}(\rho v_i \, dV) \Delta t$

total " " " in V : $\int_V \frac{\partial}{\partial t}(\rho v_i) \, dV \Delta t$

T_i : surface force/unit area on dS

total " " " : $\int_S T_i \, dS$

f_i body " / " vol in dV

total " " : $\int_V f_i \, dV$

since net force = time rate of change of momentum

$$\int_V f_i \, dV + \int_S T_i \, dS = \frac{1}{\Delta t}\left\{ \int_V \frac{\partial}{\partial t}(\rho v_i)\, dV \Delta t + \int_S \rho v_i v_j n_j \, dS \Delta t \right\}$$

now $\int_S T_i = \int_S \sigma_{ji} n_j = \int_V \sigma_{ji,j}$

$$\int_S \rho v_i v_j n_j = \int_V (\rho v_i v_j)_{,j}$$

thus

$$\int_V \left[f_i + \sigma_{ji,j} - \frac{\partial}{\partial t}(\rho v_i) - (\rho v_i v_j)_{,j} \right] dV = 0$$

since V is arbitrary , [integrand] = 0

write $\frac{\partial}{\partial t}(\rho v_i) = v_i \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial t}$

and $(\rho v_i v_j)_{,j} = v_i (\rho v_j)_{,j} + \rho v_j v_{i,j}$

combining

$$f_i + \sigma_{ji,j} - v_i \frac{\partial \rho}{\partial t} - \rho \frac{\partial v_i}{\partial t} - v_i (\rho v_j)_{,j} - \rho v_j v_{i,j} = 0$$

or

$$f_i + \sigma_{ji,j} - \rho \left[\frac{\partial v_i}{\partial t} + v_j v_{i,j} \right] - v_i \underbrace{\left[\frac{\partial \rho}{\partial t} + (\rho v_j)_{,j} \right]}_{0 \text{ from continuity}} = 0$$

$\therefore$

$$\sigma_{ji,j} + f_i = \rho \underbrace{\left[\frac{\partial v_i}{\partial t} + v_{i,j} v_j \right]}_{\text{later } \frac{Dv_i}{Dt}}$$

Equations of motion

with small velocities ~ RHS $\to \rho \ddot{u}_i$

equil " $\to 0$

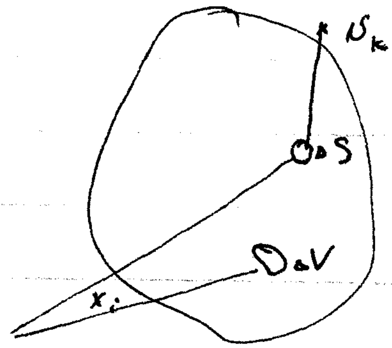
inviscid fluid or ~~equil~~ fluid at rest : pressure indep of direction

$\therefore$ isotropic

$$\therefore \quad \sigma_{ij} = (\text{const})\,\delta_{ij} \equiv -p\,\delta_{ij}$$

$$\sigma_{ij,i} = (-p\delta_{ij})_{,i} = -p_{,i}\delta_{ij} = p_{,j}$$

Moment of momentum



mass efflux in Δt : $\rho v_n \Delta t\, \Delta S$

momentum " : $\rho v_n \Delta t\, \Delta S\, v_k$

moment of " " " : $e_{ijk}\, x_j\, \rho v_n \Delta t\, \Delta S\, v_k$

mom in ΔV : $\rho\, dV\, v_k$

chg in " : $\frac{\partial}{\partial t}\left(\rho\, dV v_k\right) \Delta t$

moment of " " " : $e_{ijk}\, x_j \frac{\partial}{\partial t}\left(\rho\, dV v_k\right) \Delta t$

moment of body force in ΔV $\quad e_{ijk}\, x_j\, f_k$

" " " V $\quad \int_V e_{ijk}\, x_j\, f_k$

" " surface " in ΔS $\quad e_{ijk}\, x_j\, T_k = e_{ijk}\, x_j\, \sigma_{\ell k} n_\ell$

" " " S $\quad \int_S e_{ijk}\, x_j\, \sigma_{\ell k} n_\ell \, dS$

since net moment of forces = time rate of change of moment of momentum

$$\frac{1}{\Delta t}\left\{\int_S e_{ijk}\, x_j\, \rho v_\ell n_\ell v_k\, \Delta t\, \Delta S + \int_V e_{ijk}\, x_j \frac{\partial}{\partial t}(\rho v_k)\, \Delta t\, dV\right\}$$

$$= \int_S e_{ijk}\, x_j\, \sigma_{\ell k} n_\ell\, dS + \int_V e_{ijk}\, x_j f_k\, dV$$

$$\int_V \left[(e_{ijk}\, x_j \rho v_\ell v_k)_{,\ell} + e_{ijk}\, x_j \frac{\partial}{\partial t}(\rho v_k) - (e_{ijk}\, x_j\, \sigma_{\ell k})_{,\ell} - e_{ijk} x_j f_k \right] dV = 0$$

via thm integrand $= 0$

$$e_{ijk}\, x_{j,\ell}\, \rho v_\ell v_k + e_{ijk}\, x_j (\rho v_\ell v_k)_{,\ell} + e_{ijk}\, x_j \frac{\partial}{\partial t}(\rho v_k)$$

$$- e_{ijk}\, x_{j,\ell}\, \sigma_{\ell k} - e_{ijk} x_j\, \sigma_{\ell k,\ell} - e_{ijk} x_j f_k = 0$$

rearrange:

note $x_{j,\ell} = \delta_{j\ell}$

$$e_{ijk} \quad x_j \left[(\rho v_\ell v_k)_{,\ell} + \frac{\partial}{\partial t}(\rho v_k) - \sigma_{\ell k,\ell} - f_k \right]$$

$$+ e_{ijk} \left[\rho v_j v_k - \sigma_{jk} \right] \qquad = 0$$

the first [ ] = 0 via momentum eqn

writing what is left:

$$e_{ijk} \rho v_j v_k - e_{ijk} \sigma_{jk} = 0$$

now, since v_j a vector, $v_i v_j$ a sym. tensor

$$\therefore \quad e_{ijk} v_j v_k = 0$$

we are left with $\quad e_{ijk} \sigma_{jk} = 0$

$$\Rightarrow \sigma_{jk} = \sigma_{kj}$$

thus cons of mom of mom is satisfied
if stress tensor is symmetric.

If we consider (after Mindlin) "couple stresses"

say stress couple (tensor) : c_{ij}

surface stress couple (vector) : T_i

body couple : g_i

to the left side we must add

$$\int_V g_i \quad \text{and}$$

$$\int_S T_i = \int_S c_{ij} n_j = \int_V c_{ij,j}$$

whence the eqn reduces to

$$c_{ij,j} + g_i = e_{ijk} \sigma_{jk}$$

from which stress tensor σ_{jk} is not necessarily symmetric

we will not pursue this — we merely point it out