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TUTORIAL NOTE*
ON THE IDENTIFICATION OF LINEAR MECHANICAL SYSTEMS

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Abstract—General purpose controllers or controllers commanding systems which operate at varying conditions need a system identification routine to obtain a model for the response of the system so that it can adapt itself accordingly. Most such mechanical systems are composed of masses moving under the action of position and velocity dependent forces and hence can be modeled by second order linear differential equations. This paper describes how to obtain a simpler mathematical response model in the form of a linear difference equation for a second order linear system. Coefficients of the equation are calculated by using the least squares technique to minimize the error between the discrete position data from the system resulting when actuated by a pseudo-random binary command signal and what the model generates. An analytical solution obtained by the z-transformation of the system transfer function is also presented, to clarify the physical meaning of the coefficients. Experimental and analytical solutions for a variety of systems are presented as application examples. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Exact mathematical models of mechanical systems are derivable by Lagrange, Hamilton and Newton–Euler formulations or by energy methods. This requires all the system parameters such as masses, mass moments of inertia, stiffnesses, damping coefficients and physical dimensions explicitly. The system generally needs be dismantled into its main components where each parameter of the system is lumped for measurement. In applications where this is not possible, system identification becomes very useful, generating an empirical

*This manuscript is published for those who are new to mechatronics, especially those who are interested in system identification and who use it as a tool only; and it is believed that the material presented may act as a companion to standard textbooks.

Material presented includes the analytical expressions for the coefficients of a discrete time model and their profile with discrete time; and the profile of experimentally-obtained coefficient values, the comparison of the two, and the idea and means of obtaining the parameters of the continuous time model from identification.

mathematical model for the response of the system. System identification techniques involve probing the system with a variety of control inputs and relating the output motion to the input commands. On the basis of the amount of knowledge available about the system beforehand, the system identification problem can be classified into two categories [1] as *Complete* and *Partial Identification* problems. In the former, no characteristics of the system are known beforehand and hence it is a difficult problem to solve, while in the latter, some knowledge of the characteristics of the system like linearity, bandwidth or values of the lumped parameters is known beforehand, which greatly simplifies the identification.

In system identification, a mathematical model which may represent the system is defined and an appropriately chosen command signal is applied to the actual system. The response of the system is recorded and a parameter identification of the model that best fits the experimental data obtained is made. If the system is under continuous functional operation, the normal operating data can be used. A validation test is generally necessary to see the degree of compatibility between the model and the system. The proposed model can be in the frequency or time domain, in continuous or discrete time. In time domain modeling, the *weighting function*, *difference equation* and *state variable equation* are the choices from which to select due to the identification objective and the types of input–output data [1–3]. These models are mathematically simpler than the exact motion equations of the system although not so accurate. Exact motion equations of mechanical systems are in the form of simultaneous linear or non-linear differential equations of order at least 2. The minimum number of equations to define the system dynamics is equal to the degrees of freedom and can be obtained by Lagrange's formulation [4, 5]. Simulation studies modeling the dynamics of mechanical systems like mechanisms, robot manipulators, etc. are numerous [6]. One great difficulty in using motion equations is in their solution. Realistic equations covering all the related non-linearities like those due to the mechanism kinematics, backlash, Coulomb friction and magnetic hysteresis have no closed form solutions. Numerical solutions yielding the profile of the motion take too long a time and are generally not suitable for real time solution requirements like *model referencing* in control strategies. Early works on discretization were emphasized in the early 1970s in efforts to bring approximate solutions to the differential motion equations. As presented in the well established textbook of Cadzow [7] published in 1973, the derivatives can be represented in the form of geometric slopes of the functions at the end of each discrete time step and be substituted in place of the derivative terms. Discrete time equations resulting from similar analyses have shown that the response of systems can be represented in discrete time equations, which have then been the basis of system identification. Analog computer implementations of the equations are capable of producing real time solutions but they are not very accurate. Further, analog computer patching is a tedious task and the number of arithmetic-making modules is always limited [8]. On the contrary, the mathematics of the simple empirical models are so simple that they can be implemented digitally in or near real time for most applications. This paper aims to bring an explanation to one such mathematical model known as the *linear difference model* with application examples.

2. REPRESENTATION OF A DYNAMIC SYSTEM BY THE LINEAR DIFFERENCE EQUATION

A linear, time invariant discrete system with one input $u(k)$ and one output $y(k)$ can be characterized by a general nth order difference equation as

$$y(k)+a_1y(k-1)+\ldots+a_ny(k-n)=b_0u(k)+b_1u(k-1)+\ldots+b_nu(k-n) \quad (1)$$

or

$$y(k)+\sum_{j=1}^{n} a_jy(k-j)=\sum_{j=0}^{n} b_ju(k-j), \quad (2)$$

where k is the integer index counting the discrete time steps and a_j and b_j are constant coefficients. The independent variable for Eqns (1) and (2) is k, standing for discrete time, and the real time difference between two successive commands and similarly two response recordings is T. Order n of the equation is arbitrary. Normally accuracy and hence the reliability of the equation increases with n, but this increases the computation time of $y(k)$. In the work reported here n is taken as 2 in the modeling of second order linear mechanical systems where input is a force and output a displacement. Equation (1) for this problem simply indicates that the position y of the lumped mass at the end of the kth discrete time interval of duration T is in direct proportion to the two previous positions that have occurred 1 and 2 discrete time durations ago and with the quantities of force that have been applied 1 and 2 discrete time durations ago and the currently applied force. Coefficients of Eqn (1) are the proportionality or influence coefficients which certainly are related with the lumped parameters of the mechanical system modeled. Generation of magnitudes of these coefficients is called the *model identification*. One parameter in model identification, implicit in Eqn (1), is the *sampling interval* T. It is a common observation that the coefficients of this equation vary with T. Cadzow [7] has defined the coefficients of a first order, discrete time model to describe a second order, linear system as functions of the system lumped parameters and the discrete time. Although theoretically they appear as well-behaving functions definable for all values of T, Cadzow has indicated that coefficients generated in shorter sampling times model a response which better fits the actual result. Selection of T, the sampling interval, in system identification has been suggested to be as small as possible. This rule of thumb appears in the book edited by Mehra and Lainiotis [9], together with some more advanced concepts on system identification. Davies [10] has also indicated that an arbitrary selection of T does not generate any errors theoretically but if the sampled signal has frequency components that are higher than half the sampling frequency, aliasing in measurement will impose application errors. This is generally known as the Shannon criterion [11]. Shannon's criterion effectively keeps the control engineer on the safe side; however, analyses on what the coefficient values must be and what they appear as in identification are very scarce. This argument becomes especially important in the generation of system lumped parameters through identification. Erroneous estimations on model coefficients will certainly impose errors on the calculated values of the system parameters. In most identification applications though, the control engineer seeks a simple and reliable model disregarding the actual values of the lumped system parameters. In cases where the system under consideration is excited with frequencies beyond the fundamental, or with impulsive forces, a numerical integration of the differential motion equations becomes a better choice in visualizing a realistic response. Further, the numerical values of the coefficients of a discrete time model are less meaningful than the lumped parameters like the mass, stiffness and damping coefficient. Söderström [12] indicates that this point has not been included in his well-structured textbook, one of the most important recent publications on system identification.

3. MODEL IDENTIFICATION BY LEAST SQUARES

Equation (1) contains $(2n+1)$ coefficients and hence they can be calculated by the simultaneous solution of this many equations. Results of such an operation will not generally be so accurate. The usual practice therefore is to collect greater numbers of input and output data and apply the *least squares* technique to minimize the error between the actual discrete data output collected and what the proposed model generates. The real system can be actuated by a variety of inputs. Step input is one type which is widely preferred and hence, in the present work, the system is actuated by a *pseudo-random binary signal*, abbreviated as *PRBS*. This is the practical white noise that can easily be generated by a digital computer. White noise is a desirable input signal especially in weighting sequence estimation which reduces the related mathematics in the cross-correlation between the input and output data, enabling generation of the weighting function samples without deconvolution. Davies has explained the properties and generation of *Binary Maximum Length Sequences* in detail in his book published in 1970 [10]. The *input–output vector* $x(i)$ containing $(2n+1)$ elements is defined as

$$x(i) = [-y(k-1), \ldots, -y(k-n), u(k), \ldots, u(k-n)]^T, \tag{3}$$

where i is a counter. The *parameter vector* θ containing $(2n+1)$ elements is defined as

$$\theta = [a_1, a_2, \ldots a_n, b_0, b_1, \ldots b_n]^T. \tag{4}$$

Then, the *least squares equation* becomes

$$y(i) = x^T(i)\theta + e(i), \tag{5}$$

where $e(i)$ is the *vector of errors*. For improving accuracy, a large number N of data for input and output are collected. Substitution of them generates a system of n equations, which can be written as

$$Y = X\theta + E, \tag{6}$$

where

$$Y = [y(n+1), y(n+2), \ldots, y(n+N)]^T \tag{7}$$

$$E = [e(n+1), e(n+2), \ldots, e(n+N)]^T \tag{8}$$

and

$$X = \begin{vmatrix} -y(n) & \ldots & -y(1) & u(n+1) & \ldots & u(1) \\ -y(n+1) & \ldots & -y(2) & u(n+2) & \ldots & u(2) \\ -y(n+2) & \ldots & -y(3) & u(n+3) & \ldots & u(3) \\ . & \ldots & . & . & \ldots & . \\ . & \ldots & . & . & \ldots & . \\ -y(n+N-1) & \ldots & -y(N) & u(n+N) & \ldots & u(N) \end{vmatrix}. \tag{9}$$

From Eqn (6), θ can be estimated by means of least squares that minimize the error function J

$$J = \sum_{k=n+1}^{N+n} E^2(k) = E^T E = (Y - X\theta)^T (Y - X\theta). \tag{10}$$

Upon setting

$$\frac{\delta J}{\delta \theta} = 0 \tag{11}$$

the *parameter vector* $\hat{\theta}$ composed of the coefficients of Eqn (1), so constructed as to make this model fit the experimentally obtained data the best, appears as

$$\hat{\theta} = (XX^T)^{-1} X^T Y. \tag{12}$$

In the experimental work presented here, N is arbitrarily selected as 42. Normally, as N increases, the coefficient values become more reliable. An illustrative example on this can be found in Hsia's book [1] entitled *System Identification*. T in Eqns (3), (4), (5), (7), (8), (10) and (12) stands for "transpose", and is not the incremental time T.

4. THE SECOND-ORDER SYSTEM

The simplest mechanical system, as seen in Fig. 1a, can have an inertia element m movable in a single coordinate $y(t)$ under the effect of position and velocity dependent forces and an externally applied arbitrary actuation $u(t)$. Position dependent forces can be represented by the force of a spring with stiffness k and the velocity dependent forces by the force of a dashpot of coefficient c. A great many realistic mechanical systems can effectively be assumed to have this format. Position dependent forces may include the proportional components of the actuation forces generated by PD controllers. Similarly the velocity dependent forces may include the *Coriolis* and *centrifugal* forces, if any, forces due to viscous damping and the derivative components of the actuation forces of PD controllers. The system may contain more than one inertia element each moving in a non-linear relationship with the input like a four-bar mechanism, but may still be considered linear if it is functional within a small part of its operation range. Therefore even though the system of Fig. 1a is quite simple, it is fundamental and didactic in understanding and appreciation of system identification.

The differential equation of motion of the system shown in Fig. 1a is

$$m\ddot{y} + c\dot{y} + ky = u(t). \tag{13}$$

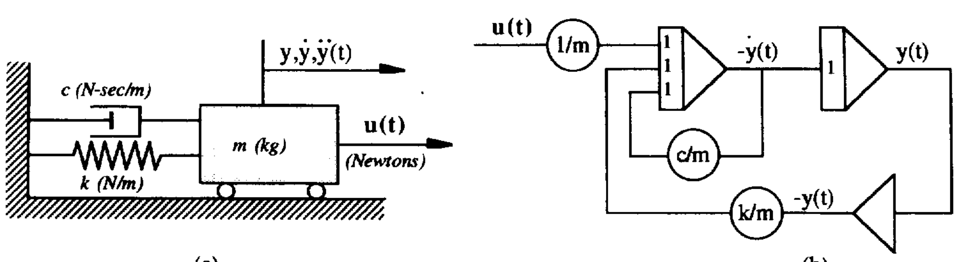


Fig. 1. (a) A simple, single degree of freedom, second order, linear mechanical system. (b) Analog simulation circuit of the same system. Time constants of the integrators are 1 sec and initial conditions are zero.

This equation in the time domain can be very complicated for a multi-body system like a mechanism and will probably have no closed-form solution. Representation of the same input–output relationship in the frequency domain has been regarded as easier to solve and also to understand. The transfer function $G(s)$, derivable from Eqn (13), defined as the ratio of system response to input in the frequency domain, is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1/m}{s^2 + (c/m)s + k/m}. \tag{14}$$

s is an indication for a function or signal to have different values at different frequencies. Equations (13) and (14) are continuous functions. Earlier techniques of estimating the lumped parameters in Eqn (14) consider their step, frequency or impulsive response in continuous time. Related examples can be found in Davies' book [10].

With the advent of digital computers, the speed of input–output processes has increased to values as high as 1 kHz and with this much computational and I/O speed computers are starting to be placed inside the control loop. Instead of differencing the analog command signal of the computer and the analog feedback signal from the system at an operational amplifier, feedback signals are directly coupled to the control computer such that the computer *reads* the state of the system and determines the magnitude of the control signal to be sent. In this, it can make use of intricate control algorithms [13] like signal filtering, compensation of steady-state errors, modification of gains and making up for time lag, instead of simply calculating the difference between the input and output variables. A digital computer is interfaced with the control system through an I/O board, which does the necessary data conversions and handshake. Analog data output is not continuous and a finite time duration elapses between two successive data outputs. Similarly data input is not continuous. The I/O device first holds the instantaneous value of the analog signal to read and conversion of this signal to digital takes a finite duration of time. This basic phenomenon compelled the continuous models in the time or frequency domain to be represented in discrete time and the tool for that is the *z transformation*. The behavior of discrete positions of second order linear systems was observed to be in the form of Eqn (1) as early as 1942 [14]. For discretization the continuous model should be put into a *sampling and holding at zero order form*. The transfer function of the *zero order hold operator* is

$$G_0(s) = \frac{1}{s} - \frac{e^{-sT}}{s} \tag{15}$$

in the s domain. The z transformation of the transfer function, pre-multiplied by the zero-order hold operator, hence yields a discrete response model in the same format as that of Eqns (1) and (2). Tables of conversion from s to z domain can be found in related textbooks. An earlier reference to obtaining discrete time models by z transformation can be found in the books of Eykhoff [15] and Cadzow [7], but without any emphasis on the practical problems of aliasing errors and estimation of lumped system parameters via identification. In the following sections, second order systems of various characteristics are analyzed and discrete time models are presented.

5. PROCEDURE AND PURPOSE OF THE EXPERIMENTAL WORK

The system under consideration is an analog simulation as shown in Fig. 1b, patched on a Comdyna GP-6 analog computer. Any non-linearity in the identified real system may affect the magnitudes of the coefficients and hence the discrepancies between the theoretical and experimentally obtained coefficient values will contain components due to non-linearities as well as aliasing. To isolate and present the effect of aliasing errors only, a linear analog computer simulation is used instead of a real rig. For control, it is connected to a Macintosh FX II computer via a 12 bit *MacADIOS* (*Macintosh Analog–Digital Input–Output System*) interface card. The analog circuit is actuated by a *PRBS* (*Pseudo-Random Binary Signal*) of 0.5 V amplitude. This corresponds to a 0.5 N force on the mass. Mass and spring constants are unity, hence the coefficient potentiometers on the position feedback and actuation force lines are set to unity. *PRBS* is a continuous set of step commands produced at T-second intervals. Polarity of the steps is random and amplitude arbitrary, but a constant; 0.5 V is a moderate value which does not cause the amplifiers of the analog simulation to saturate. Discrete time interval T is also arbitrary. However, to obtain reasonable results, the applicable range for T can have some limits. Although system identification is a well known subject, publications presenting its technology are very scarce. The purpose of the experimental work presented in this paper has been therefore to display and experimentally verify some technological aspects of the following items.

(1) To present an analytical solution for the discrete time model using z transformations.
(2) To repeat the cases handled in (1) experimentally to verify the correctness of the analytical solution.
(3) To repeat the cases handled in (1) and (2) at various T values, to see the effect of T on the coefficients of the discrete time model and suggest a sense of magnitude for the errors imposed on coefficient values due to aliasing.

6. APPLICATION EXAMPLES

Systems of various damping levels are examined as follows.

6.1. Undamped system

Substitution of zero for the damping coefficient c in Eqn (14) yields the transfer function of an undamped system as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1/m}{s^2 + k/m}. \tag{16}$$

The transfer function in z domain can be obtained by z transforming the transfer function operated on by the *zero order hold* as

$$G(z) = z[G_0(s) \cdot G(s)] = z\left(\frac{1 - e^{-sT}}{s} \frac{1/m}{s^2 + \omega_n^2}\right), \tag{17}$$

where ω_n is the natural frequency of the system equal to $\sqrt{k/m}$. Using the linearity of z transformation

$$G(z) = \frac{1}{k} z[1 - e^{-sT}] \cdot z\left(\frac{\omega_n^2}{s(s^2 + \omega_n^2)}\right) \tag{18}$$

and

$$G(z) = \frac{1}{k}\left[\frac{z-1}{z}\right]\cdot z\left(\frac{\omega_n^2}{s(s^2+\omega_n^2)}\right). \tag{19}$$

The term to be z transformed in Eqn (19) can be converted into the forms available in tables by separating it into *partial fractions* as

$$G(z) = \frac{1}{k}\left[\frac{z-1}{z}\right]\cdot z\left(\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right). \tag{20}$$

Completion of the transformation gives

$$G(z) = \frac{1}{k}\left[\frac{z-1}{z}\right]\cdot\left(\frac{z}{z-1} - \frac{z(z-\cos\omega_n T)}{z^2-2z\cos(\omega_n T)+1}\right), \tag{21}$$

and with further maths

$$G(z) = \frac{Y(z)}{U(z)} = \frac{z(1-\cos\omega_n T)/k+(1-\cos\omega_n T)/k}{z^2-2z\cos(\omega_n T)+1}. \tag{22}$$

Cross-multiplication of Eqn (22) gives

$$z^2 Y(z)+(-2\cos\omega_n T)z^1 Y(z)+z^0 Y(z) = (1-\cos\omega_n T)/kz^1 U(z)+(1-\cos\omega_n T)/kz^0 U(z). \tag{23}$$

This equation is in the same format as Eqn (1), which for $n = 2$ takes the following form:

$$y(k)+a_1 y(k-1)+a_2 y(k-2) = b_0 u(k)+b_1 u(k-1)+b_2 u(k-2). \tag{24}$$

Equating the coefficients of the corresponding terms in Eqns (23) and (24), the coefficients of the discrete time model are obtained as

$$a_1 = -2\cos(\omega_n T),\quad a_2 = 1 \tag{25}$$

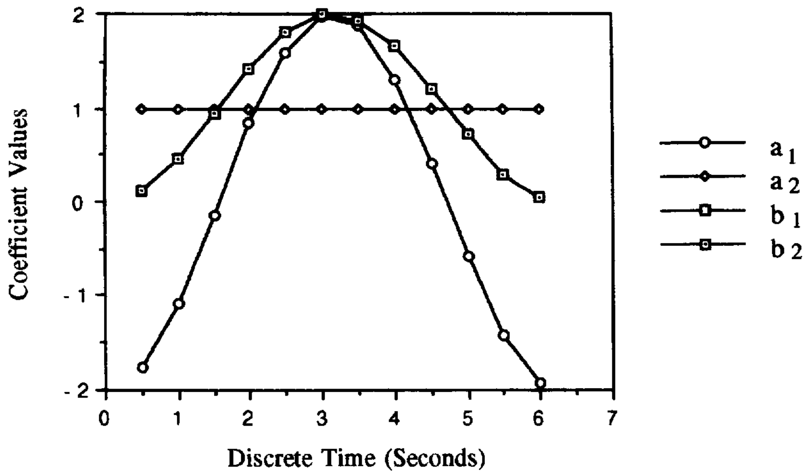
$$b_0 = 0,\quad b_1 = b_2 = \frac{1-\cos(\omega_n T)}{k}.$$

Powers of z imply the relative positions of the associated terms on the *time axis*. For the discrete time model of $n = 2$, z^2 terms correspond to the current time, z^1 terms correspond to the time T seconds before and z^0 terms correspond to the time $2T$ seconds before the current time.

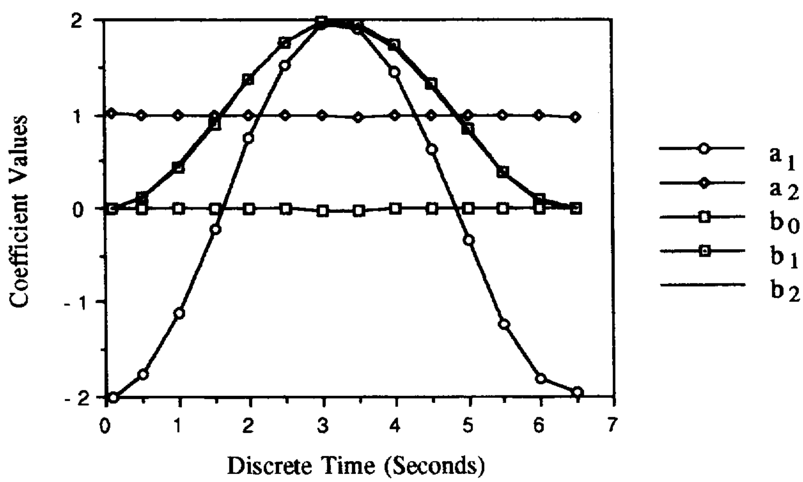
For the system under consideration, $k = 1$ N/m, $m = 1$ kg and $\omega_n = 1$ rad/sec. Therefore, once T is defined arbitrarily, the coefficients can be calculated from Eqn (25). Figure 2a shows the variation of these coefficients with T. At one extreme, when T takes the value of zero, the coefficients become $b_0 = b_1 = b_2 = 0$, $a_1 = -2$, $a_2 = 1$. All position measurements are taken at the same instant and hence have the same magnitude. Equation (24) becomes

$$y(k)-2y(k-1)+y(k-2) = 0, \tag{26}$$

which is true, but trivial. Coefficients a_2 and b_0 are constant for all values of T. The remainder, b_1, b_2 and a_1, vary sinusoidally with the system natural frequency. At the other extreme, when T takes the value of $2\pi/\omega_n$, the position at each discrete time will be the same, and hence Eqn (26) again holds true, displaying a true but trivial solution. Normally the frequency of the excitation and hence the steady state response should be much less than that of the transient response. The mathematical definitions of the coefficients given



(a)



(b)

Fig. 2. Coefficients of a second order discrete time model for a second order, linear undamped system as a function of discrete time T. (a) Results from a theoretical analysis. (b) Experimentally obtained values. Mass of the system is 1 kg and spring stiffness 1 N/m.

in Eqn (25) are continuous for all values of T. This may theoretically be true, but in reality, measurements must be taken frequently enough so as not to lose the details of the response profile. As a useful rule of thumb, Shannon indicates that sampling must be done at a rate five to 10 times the highest frequency thought to be present if aliasing problems are to be avoided [13]. Minimum allowable sampling frequency is defined in [11] as

$$\omega_s = 2\omega_n \tag{27}$$

for the undamped second order system. So, for the problem under consideration, the parts of the curves in Fig. 2 after $T = \pi$ are practically impossible to use.

For cross-checking the values of the theoretically obtained coefficients, the analog simulation shown in Fig. 1b is actuated by a 0.5 V *PRBS* of various T values. A 0.5 V $u(t)$ tends to displace the mass 0.5 m rightward from its neutral position and a -0.5 V tends to displace it 0.5 m leftward. Immediately after a random step voltage is output, the position $y(t)$ of the body is recorded. Forty-four such position recordings are taken within a total duration of $44T$ seconds for $N = 42$ and the elements of the parameter vector $\hat{\theta}$ are calculated using Eqns (9) and (12). The elements of the parameter vector, which are the coefficients of Eqn (24) when plotted against time, produce the curves shown in Fig. 2b. When compared, curves in Fig. 2a and b show a good fit. Experiments are carried out up to $T = 2\pi/\omega_n$ sec and coefficients are calculated without any difficulty. Again the problem of *how much of these curves can reliably be used* exists and the answer to that comes from Shannon's criterion.

6.2. *Underdamped system*

A damped mechanical system contains inertia, stiffness and damping elements and its transfer function is given by Eqn (14). The roots of its denominator are

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}. \tag{28}$$

In underdamped systems, restoring forces are more predominant than damping forces, i.e. $c^2/4m^2$ is less than k/m and the roots given by Eqn (28) are imaginary. Upon excitation, such a system tends to vibrate with amplitude reducing with time. The frequency of natural oscillations changes to

$$\omega_d = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}. \tag{29}$$

Substituting the *frequency of damped oscillations* into the expression for the roots

$$s_{1,2} = b \pm i\omega_d, \tag{30}$$

where $b = -c/2m$, also a system parameter. Discrete time representation of the system response is the transfer function in z domain and is obtained by z transforming the transfer function in s domain operated on by the *zero order hold* as

$$G(z) = z[G_0(s) \cdot G(s)] = z\left(\frac{1-e^{-sT}}{s}\,\frac{1/m}{s(s+b)^2+\omega_d^2}\right). \tag{31}$$

The term to be z transformed in Eqn (31) can be converted into the forms available in tables by separation into partial fractions as

$$G(z) = \frac{A}{m} z[1-e^{-sT}] \cdot z\left(\frac{A}{s} + \frac{Bs+C}{(s+b)^2+\omega_d^2}\right), \tag{32}$$

where $A = 1/(b^2 + \omega_d^2)$, $B = -A$ and $C = 2bB$. With further maths

$$G(z)=\frac{A}{m}\left[\frac{z-1}{z}\right]\cdot z\left(\frac{1}{s}-\frac{s+b}{(s+b)^2+\omega_d^2}-\frac{b}{\omega_d}\frac{\omega_d}{(s+b)^2+\omega_d^2}\right). \tag{33}$$

Completion of the transformation gives

$$G(z)=\frac{Y(z)}{U(z)}=\frac{A}{m}\left[\frac{z-1}{z}\right]\left(\frac{z}{z-1}-\frac{z^2-ze^{-bT}\cos\omega_d T}{z^2-2ze^{-bT}\cos\omega_d T+e^{-2bT}}-\frac{b}{\omega_d}\cdot\frac{ze^{-bT}\sin\omega_d T}{z^2-2ze^{-bT}\cos\omega_d T+e^{-2bT}}\right). \tag{34}$$

Cross-multiplication of Eqn (34) gives

$$z^2Y(z)+[-2e^{-bT}\cos\omega_d T]z^1Y(z)+e^{-2bT}z^0Y(z)=\frac{A}{m}\left[1-e^{-bT}\left(\cos\omega_d T+\frac{b}{\omega_d}\sin\omega_d T\right)\right]\times z^1U(z)+\frac{A}{m}\left[e^{-bT}\left(1-\cos\omega_d T+\frac{b}{\omega_d}\sin\omega_d T\right)\right]z^0U(z). \tag{35}$$

This equation is in the same format as that of Eqn (24) and equating the coefficients of their corresponding terms, the coefficients of the discrete time model of the underdamped system appear as

$$\begin{aligned}a_1&=-2e^{-bT}\cos\omega_d T,\ a_2=e^{-2bT},\ b_0=0\\ b_1&=\frac{1}{k}\left(1-e^{-bT}\left(\cos\omega_d T+\frac{b}{\omega_d}\sin\omega_d T\right)\right)\\ b_2&=\frac{1}{k}\left(e^{-bT}\left(1-\cos\omega_d T+\frac{b}{\omega_d}\sin\omega_d T\right)\right).\end{aligned} \tag{36}$$

For the system under consideration, $m=1$ kg, $c=1$ N-sec/m, $k=1$ N/m and once the incremental time T is arbitrarily defined, numerical values of the coefficients can be calculated. The profiles of the coefficients through T as defined by Eqn (36) are shown in Fig. 3a. Coefficients are continuous for all values of T, but solution for $T=0$ is physically trivial. Equation (36) indicates that the profile of a_1 starts from the value of -2, displays an oscillation with frequency ω_d and approaches to zero in T. a_2 starts from 1 and exponentially reduces to zero in T. b_1 and b_2 both start from zero, display oscillations with frequency ω_d and decay to $1/k$ and zero respectively. A lightly damped system will respond to a command swiftly and attain the commanded position after a few transient oscillations. Therefore as the incremental time increases, the effect of previous commands and position recordings on the current position becomes negligible. At the end of a large T, the mass simply displaces equal to the static deflection of the spring and comes to rest. However, the coefficients experimentally obtained from the position recordings in response to a 0.5 V *PRBS* show a sharp fall in the profiles of a_1, a_2 and b_2 at $T=\pi/\omega_d$, as seen in Fig. 3b, whereas b_1 does not deviate from the theoretical profile. This sharp fall is related to damping coefficient c and is not observed in the experimental results for the undamped system. Normally a mechanical system should not be excited with frequencies greater than or in the vicinity of ω_d. According to Shannon's criterion indicated by Eqn (27), identification results for $T\geq\pi/\omega_d$ are not practically usable. Nevertheless, the sharp fall observed in the coefficients needs further

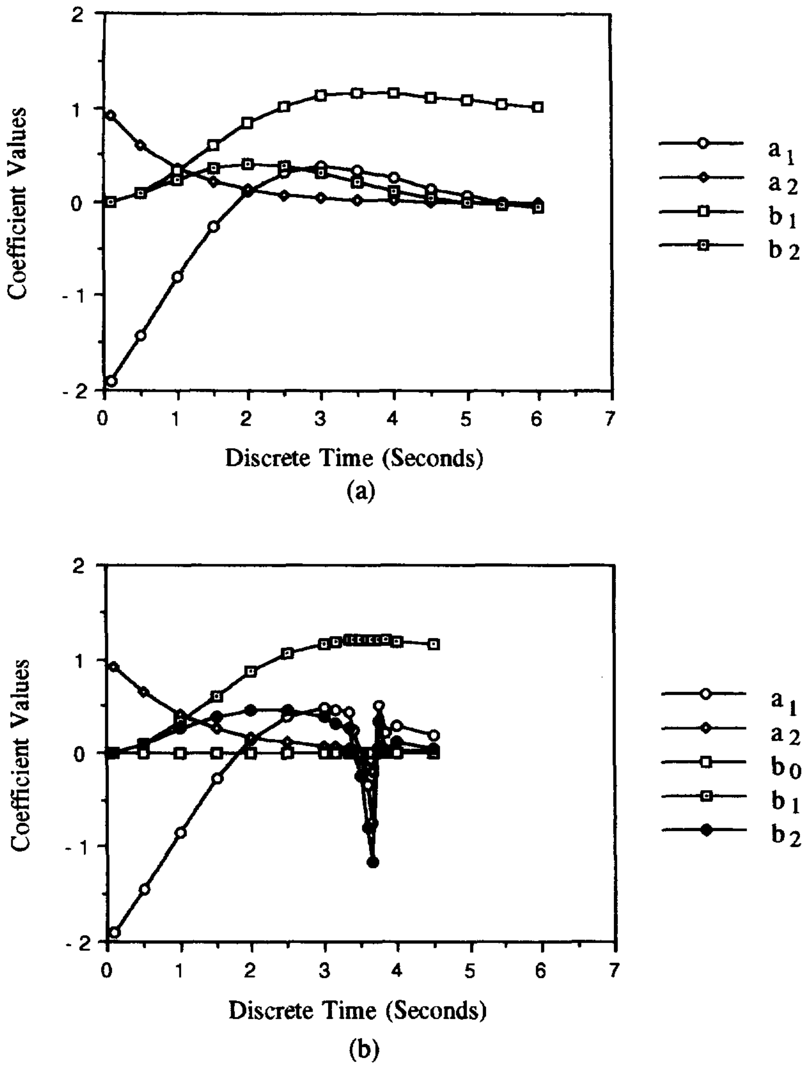


Fig. 3. Coefficients of a second order discrete time model for a second order, linear, underdamped system as a function of discrete time T. (a) Results from a theoretical analysis. (b) Experimentally obtained values. Mass of the system is 1 kg, spring stiffness is 1 N/m and damping coefficient of a dashpot parallel to the spring is 1 N-sec/m.

investigation. The trivial solution for $T = 0$ is not obtainable experimentally, hence the curves do not include these points. System parameters m, c and k are calculable from experimentally obtained coefficients by making use of Eqn (36). Profiles of the curves in Fig. 3a and b display a good fit, indicating that experimentally obtained model coefficients are accurate and reliable.

6.3. *Critically damped system*

The transfer function of a damped mechanical system is given by Eqn (14). The damping coefficient of a critically damped system is related to the other parameters of the system by

$$c_c = \sqrt{4km}, \tag{37}$$

which makes the imaginary part of the roots and hence the damped natural frequency ω_d of the system given by Eqn (30) equal to zero. Both of the roots become equal to $-c_c/2m$, which is the undamped natural frequency ω_n of the system. The transfer function in z domain can be derived as

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{k}\frac{z-1}{z}\left(\frac{z}{z-1} - \frac{\omega_n T z e^{-\omega_n T}}{(z-e^{-\omega_n T})^2} - \frac{z}{z-e^{-\omega_n T}}\right). \tag{38}$$

Re-orientation of Eqn (38) yields the discrete time model of the system in the form of Eqn (24), whose coefficients are

$$a_1 = -2e^{-\omega_n T}, \; a_2 = e^{-2\omega_n T},$$

$$b_0 = 0, \; b_1 = \frac{1-e^{-\omega_n T}(1+\omega_n T)}{k}, \; b_2 = \frac{e^{-2\omega_n T} - e^{-\omega_n T}(1-\omega_n T)}{k}. \tag{39}$$

Once T is arbitrarily defined and any two of the system parameters m, k and c_c are known, the model coefficients can be calculated. Model coefficients defined by Eqn (39) are continuous functions of the incremental time T. For a system with $m = 1$ kg, $k = 1$ N/m and $c_c = 2$ N-sec/m, the profiles of the coefficients through T are as shown in Fig. 4a. The values at $T = 0$ are physically trivial. To verify the magnitudes of the coefficients given by Eqn (39), the analog computer simulation shown in Fig. 1b, magnitude scaled to the above mentioned parameters, is subjected to system identification. The experiment is repeated for various T values to yield the profiles shown in Fig. 4b. The profiles of the theoretically obtained coefficients show the same trend as that of the underdamped system excepting the decaying oscillations, parallel to critical damping. However, the profiles of the experimentally obtained coefficients for a_1, a_2 and b_2 display a fall as seen in Fig. 4b. Experimental results for $T > \pi/\omega_n$ should not be relied upon.

6.4. *Overdamped system*

The transfer function of a damped mechanical system is given by Eqn (14). The damping coefficient of an overdamped system is related to the other parameters of the system by

$$c > \sqrt{4km}, \tag{40}$$

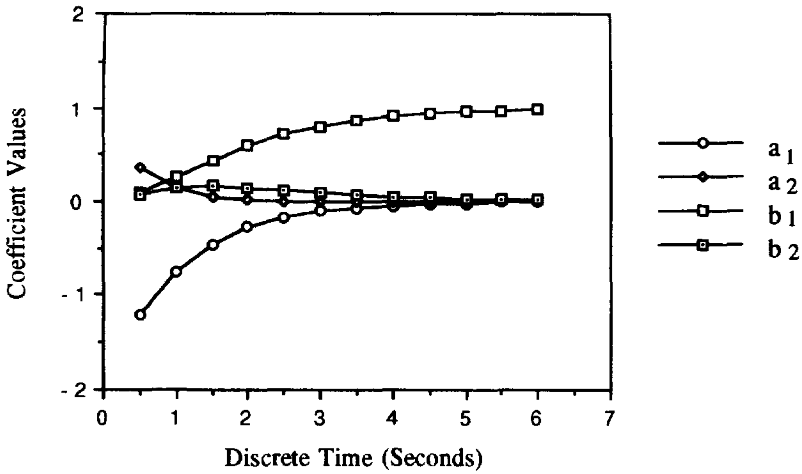
which makes the roots defined in Eqn (28) real. The transfer function in z domain can be derived as

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{m}\frac{z-1}{z}\left(\frac{z}{(z-1)pq} - \frac{z}{(z-e^{-pT})p(q-p)} + \frac{z}{(z-e^{-qT})q(q-p)}\right), \tag{41}$$

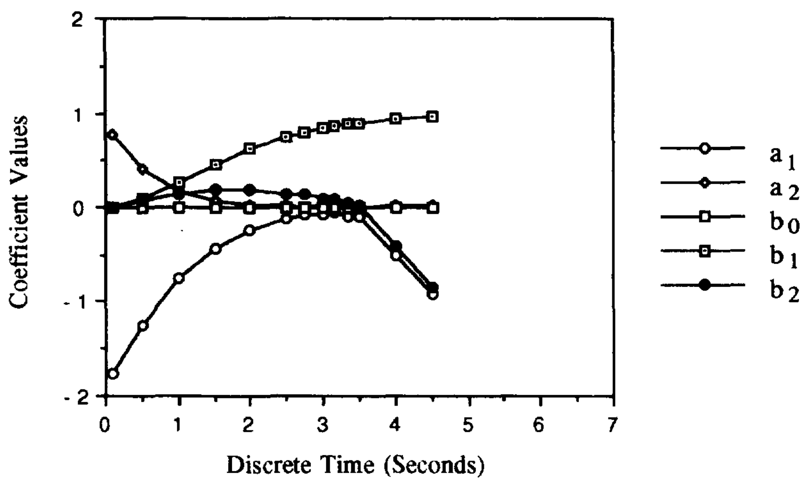
where

$$p = \frac{c}{2m} + \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} \text{ and } q = \frac{c}{2m} - \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}. \tag{42}$$

Re-orientation of Eqn (42) yields the discrete time model of the system in the form of Eqn (24), whose coefficients are



(a)



(b)

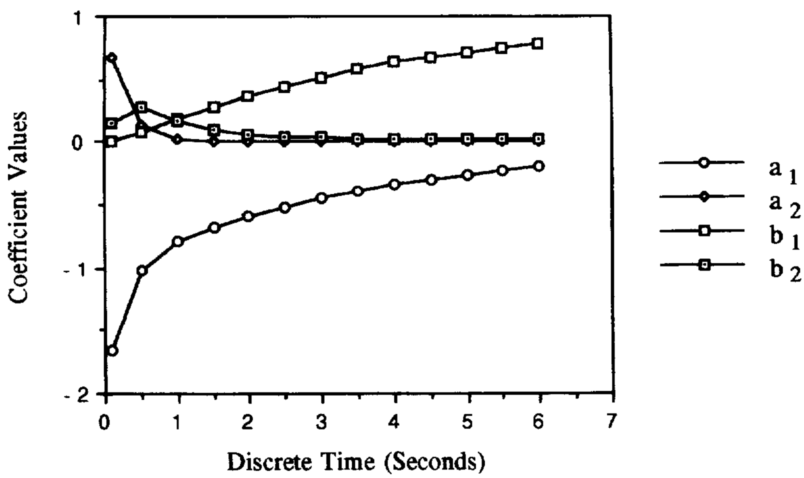
Fig. 4. Coefficients of a second order discrete time model for a second order, linear, critically damped system as a function of discrete time T. (a) Results from a theoretical analysis. (b) Experimentally obtained values. Mass of the system is 1 kg, spring stiffness is 1 N/m and damping coefficient of a dashpot parallel to the spring is 2 N-sec/m.

$$a_1 = -e^{-qT} - e^{-pT}, \quad a_2 = e^{-(p+q)T}$$

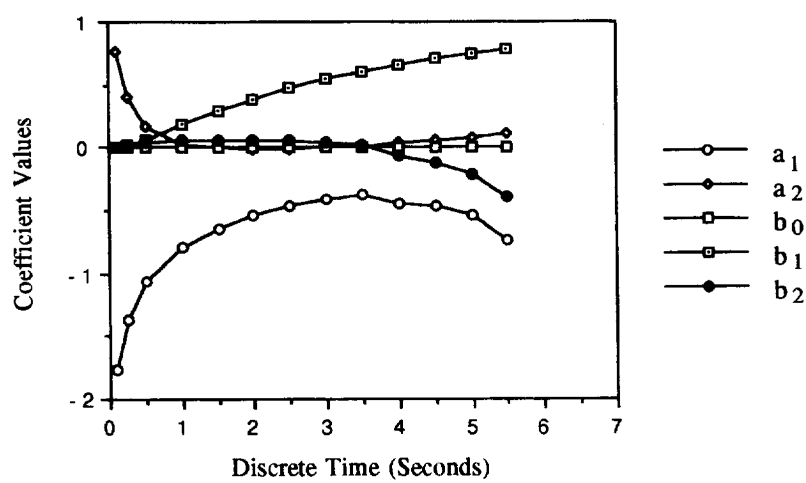
$$b_0 = 0, \quad b_1 = -\frac{e^{-qT} + e^{-pT}}{k} + \frac{1 + e^{-qT}}{mp(q-p)} - \frac{1 + e^{-pT}}{mq(q-p)}$$

$$b_2 = \frac{e^{-(p+q)T}}{k} - \frac{e^{-qT}}{mp(q-p)} + \frac{e^{-pT}}{mq(q-p)}. \tag{43}$$

Once T is arbitrarily defined and system parameters m, k and c are known, the model coefficients can be calculated. Model coefficients defined by Eqn (43) are continuous functions of the incremental time T. For a system with $m = 1$ kg, $k = 1$ N/m and $c = 4$ N-sec/m, the profiles of the coefficients through T are as shown in Fig. 5a. The values at $T = 0$ are physically trivial. To verify the magnitudes of the coefficients given by Eqn (43), the analog computer simulation shown in Fig. 1b, magnitude scaled to the above mentioned parameters, is subjected to system identification. The experiment is repeated for various T values to yield the profiles shown in Fig. 5b. The profiles of the theoretically obtained coefficients show the same trend as that of the critically damped system, but variations are more sluggish. However, the profiles of the experimentally obtained coefficients for a_1, a_2 and b_2 display a deviation from the theoretical profiles seen in Fig. 5b. Experimental results for $T > \pi/\omega_n$ should not be relied upon.



(a)



(b)

Fig. 5. Coefficients of a second order discrete time model for a second order, linear, overdamped system as a function of discrete time T. (a) Results from a theoretical analysis. (b) Experimentally obtained values. Mass of the system is 1 kg, spring stiffness is 1 N/m and damping coefficient of a dashpot parallel to the spring is 4 N-sec/m.

7. CONCLUSION

In this paper it is suggested that most second order mechanical systems can be assumed equivalent to a simple linear spring–mass–damper system. The response of such systems can be represented in the form of a discrete time model, whose coefficients can be extracted from experimentally obtained position data. A procedure for experimentally identifying the system is presented with the necessary formulations. Then it is shown that the discrete time model can be obtained by z transforming the continuous model operated on by the zero order hold. This analysis yielded a theoretical definition of model coefficients in terms of system parameters and discrete time. Solutions are given for systems of various damping levels which cannot be found in the textbooks referred to in this work. In an effort to be didactic, some intermediary steps in derivations are also given. Experimental work and theoretical analyses have brought the following conclusions.

(1) A second order linear mechanical system can be well represented by a second order discrete time model. If higher order models are used, experimentation will show that the influence of the terms at order higher than 2 is negligible.
(2) Second order discrete time models provide an estimation for the response very proximate to the real, hence they can effectively be used to provide *near-real time* solutions in simulation or control applications to replace analog models [8].
(3) System identification is generally used when the parameters of the system are unknown. Knowing the analytical expressions for the coefficients, system lumped parameters can be extracted from the experimentally obtained coefficient values. Kapucu has presented [16] test results showing the response of a mechanical system upon sinusoidal excitations of various frequencies displaying a good fit to that of a discrete time model up to the system natural frequency. Response beyond resonance is generally not required.
(4) Experimental results presented indicate that discrete time values up to π/ω_n show a good fit to the theoretical results.

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